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OPTIMIZATION OF HYBRID MARINE CELESTIAL-INERTIAL NAVIGATION SYSTEMS

by

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I. INTRODUCTION

In the past decade extensive development of self-contained marine inertial navigation devices has been undertaken in response to the need for navigation equipment which is passive in nature. The precision currently attained in the best of systems is remarkably good when the operational periods are relatively short. If the operational periods exceed one or two days, however, the navigational errors may become excessive. As a result, for extended operation the system must be periodically corrected through the use of some external position reference. One method by which the system reference may be provided is to utilize solar (or lunar) observations with a tracking radiometer. This method is unique in that it provides an all-weather observation capability. The techniques for inertial system correction using radiometric observations have not been fully exploited; accordingly, a method for system correction and the resulting error of the correction is the subject of this study. The techniques which are developed here are equally applicable to optical observation of any celestial body.

The fundamental principle underlying all inertial systems is that the vehicle's present position may be computed from its known initial position and a continuous measurement of its acceleration relative to some arbitrarily chosen frame of reference. The frame of reference is usually defined by a set of three single-degree-of-freedom gyroscopes whose input axes form an orthogonal coordinate system. Three single-degree-of-freedom accelerometers are provided to measure the vector acceleration relative to the gyro frame of reference.

A symbolic block diagram to illustrate the principles can be constructed as follows. The output, $\underline{A}$,[1] of a single-degree-of-freedom accelerometer is given by[2]

$$\underline{A} = \underline{\ddot{R}} - \underline{G} \qquad (1)$$

where $\underline{\ddot{R}}$ is the acceleration of the position vector relative to inertial space, and $\underline{G}$ is the gravitational field vector. A first order approximation for $\underline{G}$ is

$$\underline{G} = -\frac{g}{r_o}\underline{R} \qquad (2)$$

where g is the gravitation constant and r_o is the radius of the earth. Thus Equation 1 can be written as

$$\underline{\ddot{R}} = \underline{A} - \frac{g}{r_o}\underline{R} \qquad (3)$$

Symbolically, this vector equation can be implemented as shown in Figure 1.

The double integration in the loop implies that the system is basically unstable, a characteristic of all pure inertial

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[1]In this study the notation $\underline{V}$ is used to denote the vector quanity V.

[2]See, for example, McClure (4).

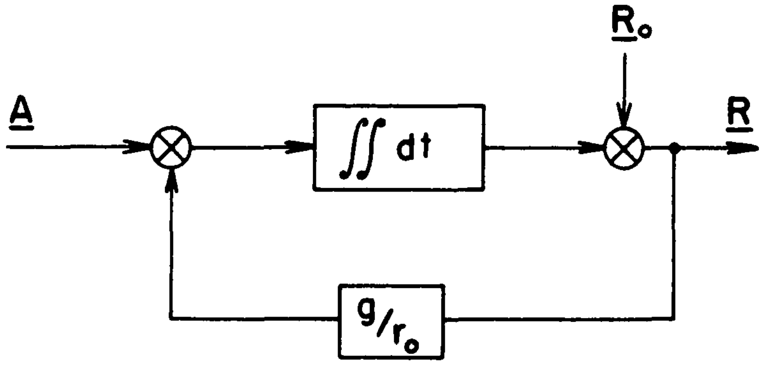

FIGURE I. SYMBOLIC REPRESENTATION OF EQUATION 3 systems. As a result, initial errors in alignment propagate sinusoidally with angular frequency g/r_0 radians per second. This period is approximately 84 minutes and is called the Schuler period. It should be noted, however, that in the absence of instrumentation errors, the system computes R without error for any dynamic input.

A variation of the pure inertial system, called "damped inertial" is used almost exclusively for marine navigation systems. This system introduces a velocity measurement, from a source external to the basic inertial system, in such a way that the response of the system is damped rather than unstable. As a result, the objectional sinusoidal propagation of initial errors is removed. Additionally, the damping bounds certain errors due to random instrument errors, (i.e., level platform tilt discussed in paragraph II.B.) which otherwise would become unbounded.

As a result of the nearly universal use of the damped inertial system for marine navigation, this system shall form the basis for the analysis which follows.

The propagation of errors in the damped inertial navigator is covered extensively in the literature.[1] As mentioned

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[1]There are three extensive reference works concerning error propagation in inertial navigation systems. Pitman (5), McClure (4), and Savant et al. (6) all have excellent treatments for the pure inertial system; Pitman (5) treats the damped inertial system.

above, some errors become bounded as a result of the damping; however, others become excessively large after extended periods of operation. As an example, the east-west position error remains unbounded even in the damped system. Accordingly, it is necessary to periodically correct the system with position information from an external source if a high order of precision is to be maintained. Optical star observations have been used successfully in this connection although they do not provide an all-weather correction capability.

Star trackers employing radiometric detection devices and operating at centimeter wavelengths are capable of providing observations of certain celestial radio sources under adverse weather conditions. Their chief limitation is that the number of sources which can be tracked is limited. For reasonable antenna sizes, tracking is restricted to the sun or moon. It may be noted, however, that the accuracy of tracking either of these sources is high, typically less than 0.5 minute of arc.

A single observation of one star is not sufficient to provide a unique system correction (specifically, position cannot be uniquely determined). However, two observations of a single star separated in time can uniquely determine position. The limiting case as the time separation approaches zero is equivalent to observation of the star's position and rate of change of position relative to the navigation coordinate system. The analytic expression for the system error vector in terms of these two parameters is derived in paragraph III.B.

The determination of the position of the star vector is limited by the tracking accuracy of the tracking equipment. In turn, this introduces an error in the computed estimate of the inertial system error. Since the star may be tracked for an extended period of time, data smoothing techniques may be applied to the raw estimate of the system error, based on the position and rate of the star vector, to provide a "best estimate" of the inertial error with a residual error smaller than that of the unprocessed estimate. Therefore, it is of interest to study the propagation of errors in the combined inertial-celestial system and to determine the "optimum data processor".

The criterion for optimization shall be that the ensemble average of the error of the estimate squared shall be a minimum at time $t_o + T$, where t_o denotes the time at the initiation of the star observation and T is the period of the observation. As discussed in Section III, the problem is fundamentally a transient one in which the estimate of the error must be obtained by an operation on finite data - finite in the sense that the period of observation is restricted. Accordingly, in this study the data processing shall be restricted to that provided by continuous linear filters with time-varying parameters which operate on the raw estimate for the finite time interval T.

The optimum filter is functionally related to the observation time T so that, in general, one cannot describe a filter which provides the best estimate of system error at any arbitrary time t. Accordingly, the estimate is optimum only at time $t_o + T$. The implication of this statement is that the inertial navigation system may be corrected at arbitrary but specific intervals of time, i.e., $t_o + T_1$, $t_o + T_1 + T_2$, $t_o + T_1 + T_2 + T_3$, ... In some applications, notably in submarines where exposure of an antenna for extended periods is operationally undesirable, it is probable that a single daily correction might be utilized. Accordingly, the expected error at the specific time $t_o + T$ is an important consideration. In other situations, especially for surface vessels, a sequence of corrections during an extended period of celestial tracking may be desirable. It should be noted, however, that a single correction with T large may suit the operational requirements better than a sequence of corrections each with small T, since in general, the error of the estimate is smaller for large T.

II. REVIEW OF INERTIAL NAVIGATOR CHARACTERISTICS

A. Choice of Coordinate System

Two general classes of inertial systems have been developed. The first class is one in which the operational time is relatively short and the vehicle paths are restricted to great circle ballistic trajectories with associated high velocities and initial accelerations. The second class is one in which there are extended operational periods and the vehicle path is restricted to the surface of the earth. In the second class the velocities and peak accelerations are appreciably less than those of the first class. Inasmuch as the environment for the two classes is different it is not surprising that the equipment associated with each of them may assume different forms. Specifically, the choice of coordinate systems, both physical and computed, is a significant design parameter from an equipment viewpoint. While in theory, all coordinate systems should be equivalent, as a practical matter specific systems are usually easier to implement in the particular coordinate system that best suits the problem specification. In this connection, for inertial navigation restricted to the surface of the earth, the choice of a locally-level coordinate system (one axis vertical) allows the system to be mechanized with only two components of the acceleration vector - the vertical component not being required. In addition, other considerations, such as control of gravity sensitive gyro biases and the availability of pitch, roll, and yaw angles, make the locally-level system very attractive for marine inertial navigation.

Accordingly, the locally-level coordinate system will be adopted as the computational coordinate system in this study.

B. Error Model for the Damped Inertial Navigator[1]

The equations of motion for a moving coordinate system are given by Equations C-7 and C-11 of Appendix C. Equation C-7 is repeated below for convenience.

$$\underline{A} - \frac{g}{r_o}\,\underline{R} = \frac{D^2\underline{R}}{dt^2} + \frac{D\underline{\omega}}{dt} \times \underline{R} + 2\underline{\omega} \times \frac{D\underline{R}}{dt} + \underline{\omega} \times (\underline{\omega} \times \underline{R}) \quad \text{(C-7)}$$

In this expression $\underline{A}$ is the accelerometer output vector, $\underline{R}$ is the position vector of the system relative to inertial space, $\underline{\omega}$ is the angular rate of the system coordinate axes, and $\frac{D}{dt}$ represents differentiation with respect to an observer rotating with the system axes. For a level coordinate system, denoted by y, with unit vectors $\underline{y}_1$, $\underline{y}_2$, and $\underline{y}_3$, north, west, and vertical respectively, Equation C-7 becomes, in column vector notation,

[1]The development of the error model in this section is an abbreviated treatment similar to a more extensive one given in Pitman (5).

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} r_o\ddot{\theta} \\ r_o\left[\Omega\dot{\theta} \sin\theta + \frac{D}{dt}(\dot{\lambda}\cos\theta)\right] \end{pmatrix} + \begin{pmatrix} r_o(\Omega-\dot{\lambda})^2 \sin\theta\cos\theta \\ r_o\dot{\theta}(\Omega-\dot{\lambda})\sin\theta \end{pmatrix} \qquad (4)$$

In Equation 4, θ is latitude, λ is longitude, Ω is the scaler magnitude of the earth's rotation rate, and A_1 and A_2 are the north and west components of the vehicle thrust acceleration vector measured relative to the y-coordinate system. The dot notation for derivatives means differentiation with respect to the system axes and is equivalent to the notation $\frac{D}{dt}$. These equations may be mechanized as shown in Figure 2.

For purposes of error analysis, it is convenient to omit the second term of Equation 4. This term is due to the centrifugal acceleration and in a typical situation, omission results in a vertical error of less than 30 minutes of arc. Therefore, its effect on the propagation of errors is negligible. We also assume that $\dot{\theta}$ is small and may be similarly omitted. It should be noted that these approximations are valid only in a study of errors, and that the system itself must be fully mechanized according to Equation 4. Further, there are other effects such as gravity anomalies and the oblateness of the earth which must normally be computed in a precision system.

Implicit in Equation 4, is a frame of reference, in this case locally level and with one axis north, in which the

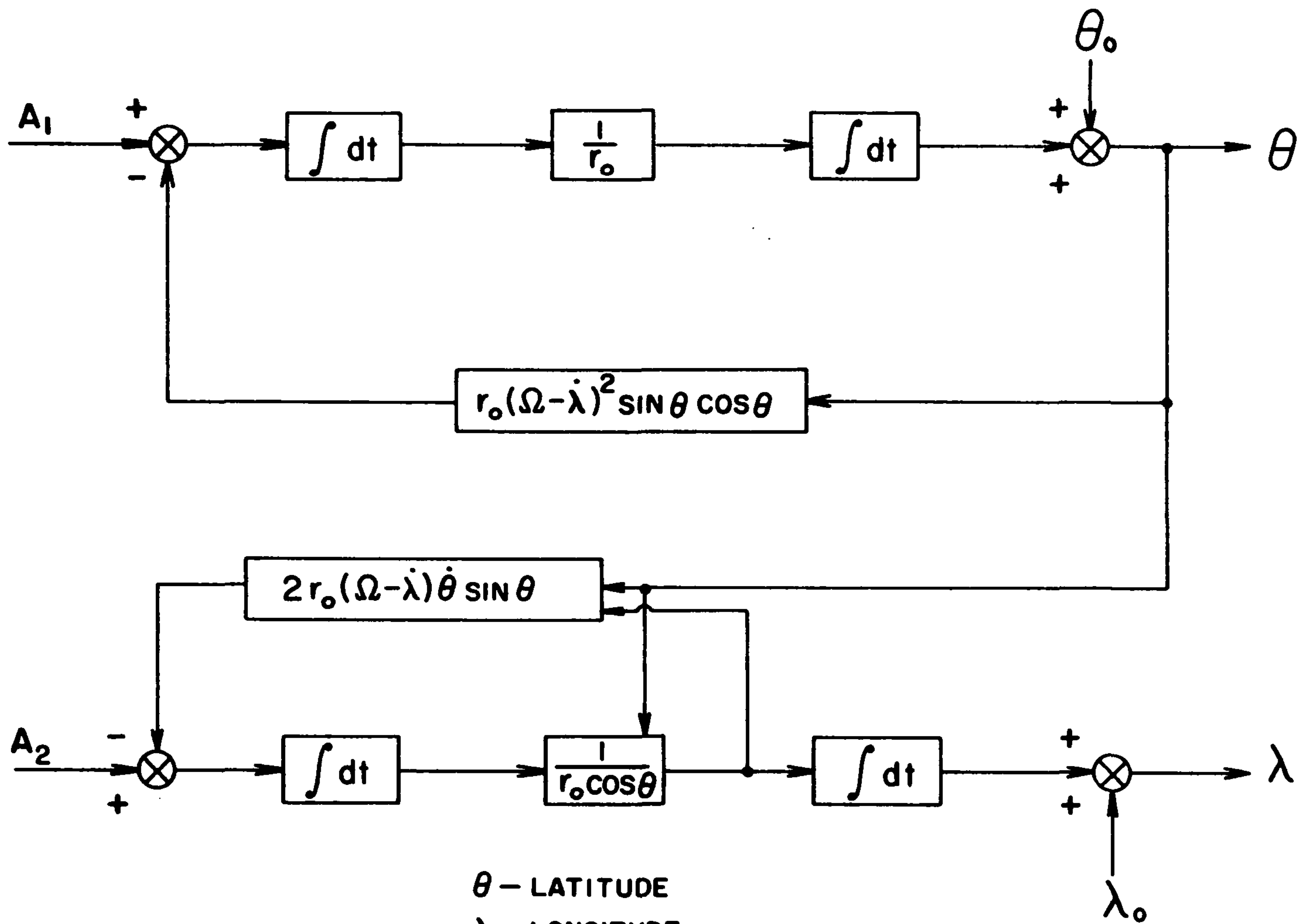

FIGURE 2. SIMPLE MODEL OF AN INERTIAL NAVIGATOR

vector $\underline{A}$ is measured. This frame is normally provided by three single-degree-of-freedom gyros whose input axes are coincident with the axes of the three (two in this case) orthogonal accelerometers. Since the gyros maintain their orientation in inertial space, they must be torqued at rates equal to the angular rate of the position vector in inertial space. For the coordinate system considered here, these rates are $(\Omega - \dot{\lambda})\cos\theta$, $\dot{\theta}$, and $(\Omega - \dot{\lambda})\sin\theta$ for the north, west, and vertical axes respectively.

Consider two sets of axes: (1) the y-coordinate system with axes $\underline{y}_1$, $\underline{y}_2$, and $\underline{y}_3$ which are north, west, and vertical respectively and with origin at the true position vector; and (2) the z-coordinate system with axes $\underline{z}_1$, $\underline{z}_2$ and $\underline{z}_3$ nominally north, west, and vertical and with origin at the true position vector, but whose actual positions are coincident with the instrument axes. The misalignment between the y- and z-coordinate systems may be expressed in terms of a misalignment vector, $\underline{\zeta}$ given by:

$$\underline{\zeta} = \zeta_1 \underline{y}_1 + \zeta_2 \underline{y}_2 + \zeta_3 \underline{y}_3 \tag{5}$$

where ζ_1 is a small rotation about the $\underline{y}_1$ axis. Specifically, the z-coordinate system may be viewed as a rotation of the y-coordinate system through the vector $\underline{\zeta}$. Thus[1]

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[1]See Appendix A.

$$\underline{z}_i = \underline{y}_i + \underline{\zeta} \times \underline{y}_i \tag{6}$$

It should be noted that for small rotations

$$\underline{z}_i \cong \underline{y}_i + \underline{\zeta} \times \underline{z}_i \tag{7}$$

If the z-system is misaligned from the true y-coordinate system by the rotation vector $\underline{\zeta}$, the components of the acceleration vector, $\underline{A}_z$, measured in the z-system, are interpreted as components in the direction of the y-system axes. As a result, the acceleration vector used by the system, $\underline{A}$, is related to the true acceleration vector by[1]

$$\underline{A}_z = \underline{A} + \underline{\zeta} \times \underline{A} \tag{8}$$

Similarly, the components of vector rotation of the z-coordinate axes are computed on the basis of the y-system axes,[2] so that the z-system rotation vector, $\underline{\omega}_z$, is related to the y-system vector, $\underline{\omega}$, by the equation

$$\underline{\omega}_z = \underline{\omega} + \underline{\zeta} \times \underline{\omega} \tag{9}$$

For small velocities relative to the earth, this may be expressed as

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[1]See Appendix A.

[2]This statement is valid provided the error δd_2 is not large. This quantity introduces an additional computational error in $\underline{\omega}$ of the form $\Omega \delta d_2 \cos \theta$ which is typically less than the components of the gyro drift rate vector, $\underline{\epsilon}$.

$$\underline{\omega}_z \cong \underline{\omega} + \underline{\zeta} \times \underline{\Omega} \tag{10}$$

where $\underline{\Omega}$ is the earth's rotation vector. Finally the gravity vector $\underline{G}$ becomes

$$\underline{G}_z = \underline{G} + \underline{\zeta} \times \underline{G} \tag{11}$$

For the level system $\underline{G} \cong - g \, \underline{Y}_3$ so that Equation 11 becomes

$$\underline{G}_z = - g \, \underline{Y}_3 - g \, \underline{\zeta} \times \underline{Y}_3 \tag{12}$$

Since the system is approximately linear, the principle of superposition permits one to draw the error diagram, shown in Figure 3, by resolution of the errors of the above vectors into their respective y-components. The components of the gyro drift rate vector, $\underline{\epsilon}$, have been added, as well as the accelerometer instrumentation errors, $\underline{\delta A}$. As discussed in Section III, the components ζ_1 and ζ_2 of the misalignment vector are typically small and have been neglected.

The above treatment is necessarily brief; for a more comprehensive discussion see Pitman (5).

In Figure 3, the distance errors along the $\underline{Y}_1$ and $\underline{Y}_2$ axes are denoted by δd_1, and δd_2 respectively. These are the errors in which we are principally interested. As mentioned earlier, it is customary to provide an external measurement of velocity which is compared with that generated by the inertial computation. The difference may be fed back to the accelerometer output to provide a damping term. This is seen by con-

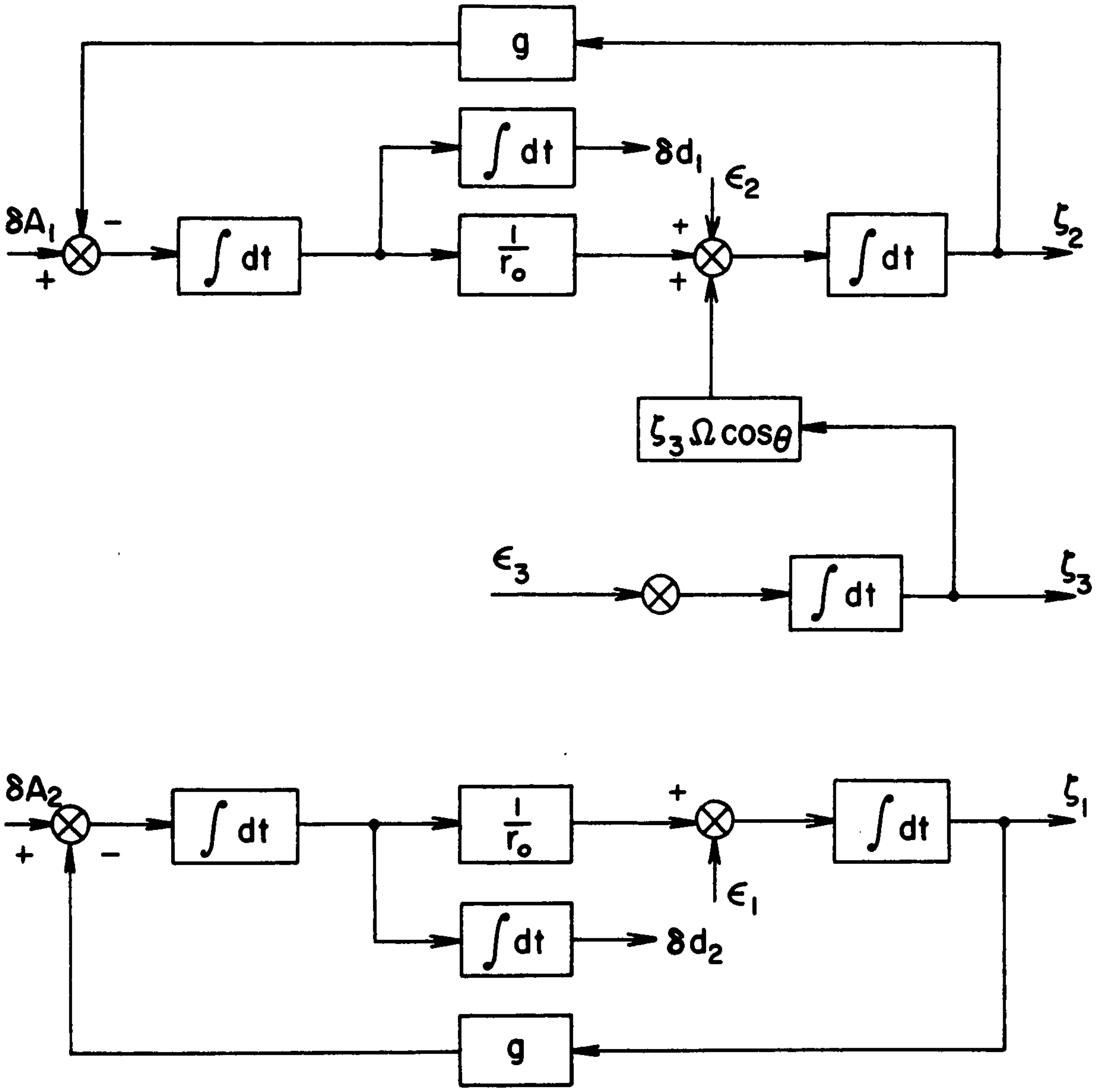

FIGURE 3. ERROR DIAGRAM FOR THE PURE INERTIAL NAVIGATOR

sidering only the velocity feedback loop. (See Figure 4.)

In Figure 4,[1] $\hat{V}$ is the externally measured velocity. The closed-loop transfer function is

$$\frac{V}{A}(s) = \frac{1}{s + k}$$

Clearly, since there is only one other integration in the loop, the loop is unconditionally stable. Thus Figure 3 may be modified to include the effects of errors in $\hat{V}$. This is shown in Figure 5.

Actually the simple gain constant k may be replaced by any frequency dependent network, provided that the system remains stable. The same is true for the simple integrations shown in Figure 2, although the system error is then dependent on the dynamics of the inputs. Nevertheless, if sufficient statistical information regarding the inputs and instrument errors is available, these transfer functions may be chosen to provide an "optimum" system performance. This problem is covered in the literature and will not be treated here.

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[1] In Figure 4, an s-plane representation, rather than the real time integral representation used earlier, has been employed to describe the transfer characteristics of the loop components. The variable s is the Laplace transform variable.

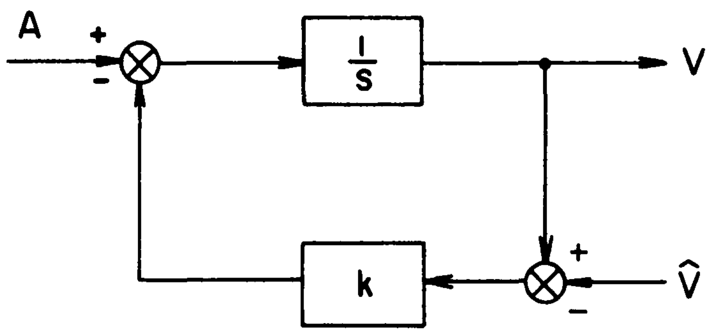

FIGURE 4. VELOCITY FEEDBACK LOOP

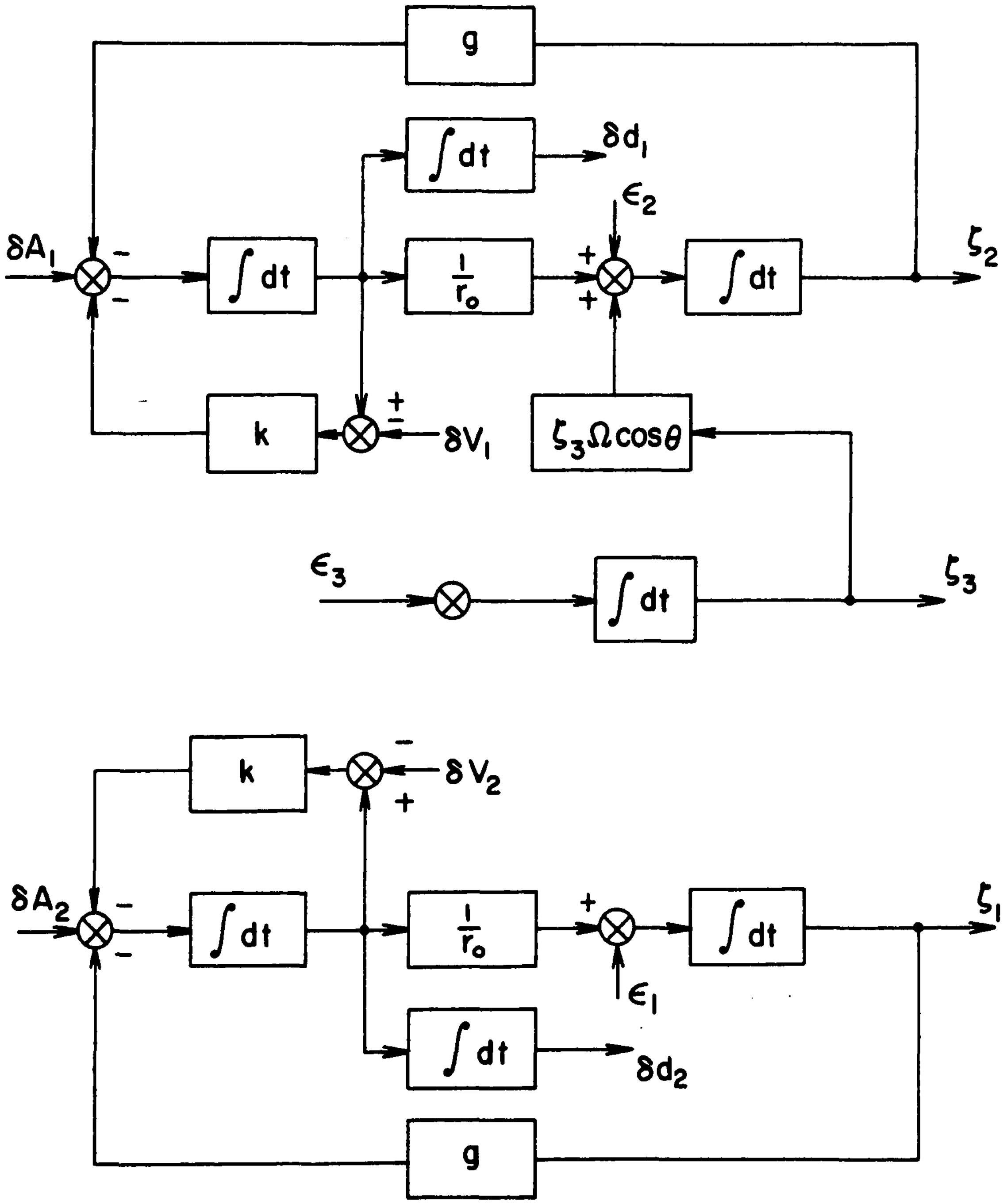

FIGURE 5. ERROR DIAGRAM FOR THE DAMPED INERTIAL NAVIGATOR

III. DETERMINATION OF AZIMUTH AND POSITION ERRORS

The salient feature of a damped inertial navigation system is that the damping, in conjunction with the gravity feedback, produces a system in which the initial level errors, due to misalignments or position errors are ultimately reduced to zero. The system level error in the steady state is essentially determined then by the instrument errors. In a precision damped inertial system, level errors are typically less than ten seconds of arc which represent a position error of less than 1500 ft. due to this source. The position errors, however, are not necessarily small even though the level errors may be. Reference to Figure 5 indicates that a first order approximation to the distance error is given by

$$\text{distance error} = r_o \int \epsilon \, dt$$

for $\zeta_1 \ll 1$. In the above relation r_o is the radius of the earth and ϵ is the gyro drift rate. Interaxis coupling is neglected in the expression above and ζ_1 and ζ_2 are assumed to be small. Accordingly, distance errors are not necessarily bounded.[1] The reader is referred to Pitman (5) for an excellent discussion of both the long-term and short-term propaga-

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[1]Actually, latitude and azimuth errors due to gyro drift rate biases are bounded - the longitude error is not bounded. Errors due to random components of gyro drift rate may not be bounded in any coordinate.

tion of errors for the damped inertial navigation system.

When the level errors, ζ_1 and ζ_2, are small, an extended observation of the relative coordinates of one (or more) star position vector is sufficient to determine an estimate of the position error of the system. This is discussed in detail below.

A. Relation of Error Angle Vectors to Azimuth and Position Errors

In the preceding material, two coordinate systems have been considered: (1) the y-coordinate system with $\underline{y}_1$, $\underline{y}_2$, $\underline{y}_3$ unit vectors north, west, and vertical respectively and with origin at the true position vector; and (2) the z-coordinate system with axes nominally north, west, and vertical with origin at the true position vector, but displaced from their nominal positions by small rotations about the $\underline{y}_i$ axes. The rotations were described in terms of a rotation vector $\underline{\zeta} = \zeta_1 \underline{y}_1 + \zeta_2 \underline{y}_2 + \zeta_3 \underline{y}_3$. The z-coordinate system represents the actual position of the instrument (gyro and accelerometer) axes.

It is convenient to introduce two additional sets of axes. The x-coordinate system axes defined by unit vectors $\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$, which are north, west, and vertical respectively with the origin at the computed position vector. The p-system axes with unit vectors $\underline{p}_1$, $\underline{p}_2$, $\underline{p}_3$, are defined to be north, west and vertical with origin at $r_o \underline{P}$ relative to the center of the earth, where $\underline{P}$ is the unit vector in the direction of the observed star (sun). It may be noted that the axis $\underline{p}_3$ is equal to $\underline{P}$.

The position of the x-coordinate system relative to the z-coordinate system may be described by a rotation vector,

$$\underline{\chi} = \chi_1 \underline{z}_1 + \chi_2 \underline{z}_2 + \chi_3 \underline{z}_3 \tag{13}$$

where χ_i represents a small rotation about the $\underline{z}_i$ axis. Thus[1]

$$\underline{x}_i = \underline{z}_i + \underline{\chi} \times \underline{z}_i \tag{14}$$

Figure 6 is a plane representation of one component of the vectors $\underline{\zeta}$ and $\underline{\chi}$, and may aid in visualizing the vector rotations; $\underline{R}$ is the true position vector, and $\underline{R} + \underline{\delta R}$ is the computed position vector. Now consider the case of $\zeta_1 \cong \zeta_2 \cong 0$. The first two components of Equation 14 become

$$\underline{x}_1 \cong \underline{y}_1 + \underline{\chi} \times \underline{y}_1 \tag{15}$$

$$\underline{x}_2 \cong \underline{y}_2 + \underline{\chi} \times \underline{y}_2 \tag{16}$$

Further

$$\chi_3 \cong - \zeta_3 \tag{17}$$

so that χ_3 represents the azimuth error. Reference to Figure

[1]See Appendix A.

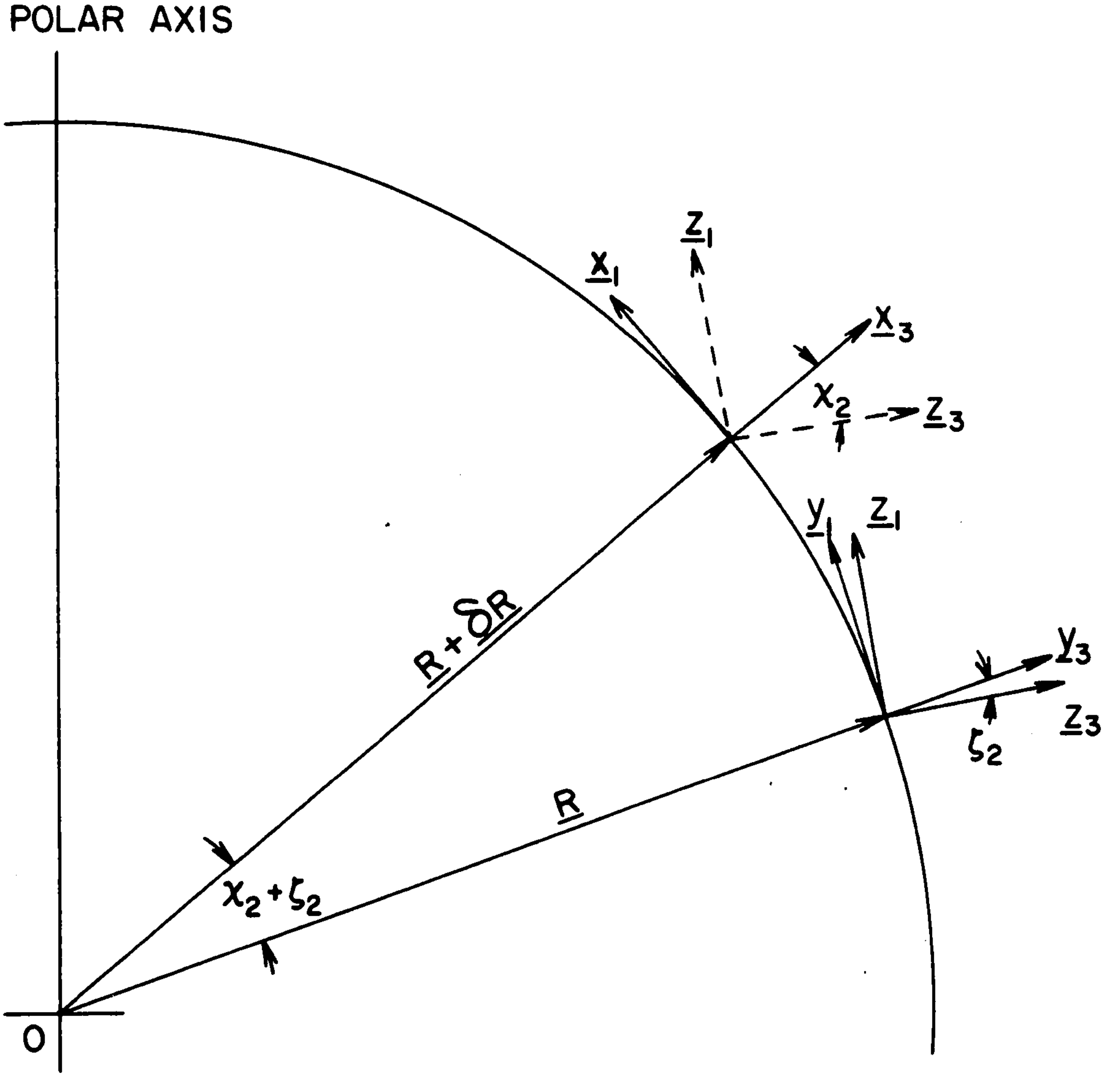

FIGURE 6. REPRESENTATION OF THE VECTOR COMPONENTS χ_2 AND ζ_2.

6 reveals that, for ζ_1 and ζ_2 small, the angle χ_2 is approximately the distance error along axis y_1 (or z_1) divided by the earth's radius, i.e.,

$$\chi_2 = \frac{y_1 \text{ distance error}}{r_o} \tag{18}$$

Similarly

$$\chi_1 = \frac{y_2 \text{ distance error}}{r_o} \tag{19}$$

Thus if the components of the vector $\underline{\chi}$ can be obtained, the azimuth and position errors may be computed.

B. Estimate of the Error Vector and the Error of the Estimate

Consider the observed star vector $\underline{P}$ which is fixed in inertial space. The components of $\underline{P}$ are tabulated relative to inertial space (or in an earth-fixed coordinate system); thus the system may compute the components of $\underline{P}$ relative to the x-coordinate system without error. Accordingly, $\underline{P}$ may be written as:

$$\underline{P} = P_{x_1}\underline{x}_1 + P_{x_2}\underline{x}_2 + P_{x_3}\underline{x}_3 \tag{20}$$

where P_{x_1} is the component of $\underline{P}$ along axis $\underline{x}_1$ and may be regarded as known. A star tracker pointing in the direction of $\underline{P}$ measures the components of $\underline{P}$ relative to the z-coordinate system. Thus the measured vector is

$$\underline{P} = P_{z_1}\underline{z}_1 + P_{z_2}\underline{z}_2 + P_{z_3}\underline{z}_3 \tag{21}$$

where the components P_{z_i} are measured quantities. The system interprets the measured components P_{z_i} as being along the axis $\underline{x}_i$, and accordingly detects a difference between the measured and computed vector $\underline{P}$. This difference, $\underline{\delta P}$, is given by

$$\underline{\delta P} = (P_{x_1} - P_{z_1})\underline{x}_1 + (P_{x_2} - P_{z_2})\underline{x}_2 + (P_{x_3} - P_{z_3})\underline{x}_3 \tag{22}$$

Another expression for $\underline{\delta P}$ may be obtained by substituting $\underline{z}_i \cong \underline{x}_i - \underline{\chi} \times \underline{x}_i$ in Equation 21 to give

$$\underline{P} = P_{z_1}(\underline{x}_1 - \underline{\chi} \times \underline{x}_1) + P_{z_2}(\underline{x}_2 - \underline{\chi} \times \underline{x}_2) + P_{z_3}(\underline{x}_3 - \underline{\chi} \times \underline{x}_3) \tag{23}$$

Subtracting Equation 23 from 20 gives

$$P_{x_1}\underline{x}_1 + P_{x_2}\underline{x}_2 + P_{x_3}\underline{x}_3 - P_{z_1}(\underline{x}_1 - \underline{\chi} \times \underline{x}_1) - P_{z_2}(\underline{x}_2 - \underline{\chi} \times \underline{x}_2) + P_{z_3}(\underline{x}_3 - \underline{\chi} \times \underline{x}_3) = 0$$

Noting the expression for $\underline{\delta P}$ in Equation 22, this reduces to

$$\underline{\delta P} = - P_{z_1}\underline{\chi} \times \underline{x}_1 - P_{z_2}\underline{\chi} \times \underline{x}_2 - P_{z_3}\underline{\chi} \times \underline{x}_3 \tag{24}$$

However, since the rotations are small, Equation 24 can be written as

$$\underline{\delta P} \cong - P_{z_1}\underline{\chi} \times \underline{z}_1 - P_{z_2}\underline{\chi} \times \underline{z}_2 - P_{z_3}\underline{\chi} \times \underline{z}_3 \tag{25}$$

Or:

$$\underline{\delta P} = - \underline{\chi} \times \underline{P} \tag{26}$$

Inasmuch as $\underline{\delta P}$ can be computed from Equation 22, Equation 26 provides a method for measurement of certain components of $\underline{\chi}$. Letting $\underline{\chi} = a\underline{\delta P} + b\underline{P} + c\underline{\delta P} \times \underline{P}$, and substituting in Equation 26 gives

$$\underline{\chi} = b\underline{P} + \underline{\delta P} \times \underline{P} \tag{27}$$

The coefficient b cannot be determined from Equation 26, so that Equation 27 does not uniquely define $\underline{\chi}$. This is to be expected since rotations about $\underline{P}$ cannot be detected. The ambiguity may be removed by observation of another star whose position vector is not colinear with $\underline{P}$. This case has been treated very briefly in Pitman (5). One may also remove the ambiguity by utilization of the earth's rotation vector and the associated rate of change of the vector $\underline{P}$ with respect to the z-coordinate system.

Specifically, using the Theorem of Coriolis, the inertial space derivative of $\underline{\chi}$ may be expressed as

$$\frac{d\underline{\chi}}{dt} = \frac{D\underline{\chi}}{dt} + \underline{\omega} \times \underline{\chi} \tag{28}$$

where d/dt represents differentiation as viewed by an observer in inertial space, D/dt represents differentiation as viewed by an observer in the x-coordinate system, and $\underline{\omega}$ is the angular rotation vector of the x-coordinate system. Differentiating Equation 27 gives another expression for $d\underline{\chi}/dt$:

$$\frac{d\underline{\chi}}{dt} = \frac{db}{dt}\,\underline{P} + \frac{d\underline{\delta P}}{dt} \times \underline{P} \tag{29}$$

In Equation 29 $d\underline{P}/dt$ has been set equal to zero since $\underline{P}$ is a constant vector in inertial space. Again, applying the Theorem of Coriolis, Equation 29 may be written as

$$\frac{d\underline{\chi}}{dt} = \frac{db}{dt}\,\underline{P} + (\dot{\underline{\delta P}} + \underline{\omega} \times \underline{\delta P}) \times \underline{P} \tag{30}$$

where the dot notation is equivalent to D/dt. Since b is a scaler,

$$\frac{db}{dt} = \frac{Db}{dt} = \dot{b} \tag{31}$$

Expansion of the triple vector product in Equation 30 gives

$$\frac{d\underline{\chi}}{dt} = \dot{b}\underline{P} + \dot{\underline{\delta P}} \times \underline{P} + (\underline{\omega}\cdot\underline{P})\underline{\delta P} - (\underline{\delta P}\cdot\underline{P})\underline{\omega} \tag{32}$$

Since $\underline{P}$ is a unit vector, $\underline{\delta P}$ is orthogonal to $\underline{P}$ so that $\underline{\delta P}\cdot\underline{P}$ is zero. Thus Equation 32 becomes

$$\frac{d\underline{\chi}}{dt} = \dot{b}\underline{P} + \dot{\underline{\delta P}} \times \underline{P} + (\underline{\omega}\cdot\underline{P})\underline{\delta P} \tag{33}$$

Substituting Equation 28 in 33 gives

$$\underline{\omega} \times \underline{\chi} = \dot{b}\underline{P} + \dot{\underline{\delta P}} \times \underline{P} + (\underline{\omega}\cdot\underline{P})\underline{\delta P} - \dot{\underline{\chi}} \tag{34}$$

Referring to Figure 6, it is seen that if ζ_1 and ζ_2 are small, the angular position errors in the direction of the $\underline{y}_1$ and $\underline{y}_2$ axes are given by χ_2 and χ_1 respectively; also $\chi_3 = -\zeta_3$ from Equation 17. If the interaxis coupling is small, Figure 5 shows that

$$\frac{D\chi_1}{dt} - \epsilon_1 = \frac{D\zeta_1}{dt} \cong 0 \qquad (35)$$

$$\frac{D\chi_2}{dt} - \epsilon_2 = \frac{D\zeta_2}{dt} \cong 0 \qquad (36)$$

$$\frac{D\chi_3}{dt} = - \frac{D\zeta_3}{dt} = \epsilon_3 \qquad (37)$$

In vector notation, Equations 35, 36, and 37 become

$$\frac{D\underline{\chi}}{dt} = \dot{\underline{\chi}} = \underline{\epsilon} \qquad (38)$$

Substituting Equation 38 in Equation 34 gives

$$\underline{\omega} \times \underline{\chi} = \dot{b}\underline{P} + \dot{\underline{\delta P}} \times \underline{P} + (\underline{\omega}\cdot\underline{P})\underline{\delta P} - \underline{\epsilon} \qquad (39)$$

Combining Equations 27 and 39

$$\underline{\omega} \times (b\underline{P} + \underline{\delta P} \times \underline{P}) = \dot{b}\underline{P} + \dot{\underline{\delta P}} \times \underline{P} + (\underline{\omega}\cdot\underline{P})\underline{\delta P} - \underline{\epsilon} \qquad (40)$$

Expanding the triple vector product gives

$$b\underline{\omega} \times \underline{P} - (\underline{\delta P}\cdot\underline{\omega})\underline{P} = \dot{b}\underline{P} + \dot{\underline{\delta P}} \times \underline{P} - \underline{\epsilon} \qquad (41)$$

Equating components in the direction of $\underline{P}$ results in

$$- \underline{\delta P}\cdot\underline{\omega} = \dot{b} - \underline{\epsilon}\cdot\underline{P} \qquad (42)$$

The remaining terms give

$$b \underline{\omega} \times \underline{P} = \dot{\underline{\delta P}} \times \underline{P} - [\underline{\epsilon} - (\underline{\epsilon}\cdot\underline{P})\underline{P}] \tag{43}$$

Forming the scaler product of Equation 43 with the vector $\underline{\omega} \times \underline{P}$ and expanding gives

$$b = \frac{1}{(\underline{\omega} \times \underline{P})^2} [\dot{\underline{\delta P}}\cdot\underline{\omega} - \underline{\epsilon}\cdot\underline{\omega} \times \underline{P}] \tag{44}$$

In Equation 44, we have used the fact that $\dot{\underline{\delta P}}\cdot \underline{P} = 0$. Substituting Equation 44 in Equation 27 gives the complete expression for $\underline{\chi}$:

$$\underline{\chi} = \frac{1}{(\underline{\omega} \times \underline{P})^2} [\dot{\underline{\delta P}}\cdot\underline{\omega} - \underline{\epsilon}\cdot\underline{\omega} \times \underline{P}] \underline{P} + \underline{\delta P} \times \underline{P} \tag{45}$$

It is seen that all quantities on the right side of Equation 45 are known or can be measured except the gyro drift rate vector $\underline{\epsilon}$. If $\underline{\epsilon}$ were known, $\underline{\chi}$ could be uniquely determined. However an estimate of $\underline{\chi}$ can be obtained by assuming $\underline{\epsilon}$ to be zero,[1] as described below.

The star tracker introduces a measurement error so that the vector represented by the pointing vector of the device is actually $\underline{P}_z + \underline{\Delta P}$ where $\underline{\Delta P}$ is orthogonal to $\underline{P}_z$. Accordingly, the estimate of $\underline{\chi}$ based on Equation 45 with $\underline{\epsilon}$ equal to zero is

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[1] $\underline{\epsilon}$ may be chosen to be any vector function. However, for typical gyros the ensemble average of $\underline{\epsilon}$ is normally zero; and thus it is chosen to be zero here.

$$\hat{\underline{\chi}} = \frac{1}{(\underline{\omega} \times \underline{P})^2}\left[(\dot{\underline{\delta P}} + \dot{\underline{\Delta P}}) \cdot \underline{\omega} \right] \underline{P} + (\underline{\delta P} + \underline{\Delta P}) \times \underline{P} \qquad (46)$$

Equation 46 is the equation which must be mechanized to form the raw estimate of $\underline{\chi}$. In implementing Equation 46, some confusion may result from the terms containing $\underline{\Delta P}$. It should be noted that $\underline{\delta P}$ computed from the measured quantities P_{z_1}, using Equation 22, actually gives $\underline{\delta P} + \underline{\Delta P}$. Hence the estimate of $\underline{\chi}$ computed by the system is $\hat{\underline{\chi}}$ given in Equation 46.

When the velocity relative to the earth is small, $\underline{\omega}$ is approximately equal to the earth's rotation vector, $\underline{\Omega}$, and Equation 46 reduces to a somewhat simpler form:

$$\hat{\underline{\chi}} = \frac{1}{\Omega^2(1 - \sin^2 d)}\left[(\dot{\underline{\delta P}} + \dot{\underline{\Delta P}}) \cdot \underline{\Omega} \right] \underline{P} + (\underline{\delta P} + \underline{\Delta P}) \times \underline{P} \qquad (47)$$

where d is the declination of the star. Similarly, Equation 45 becomes

$$\underline{\chi} = \frac{1}{\Omega^2(1 - \sin^2 d)}\left[\dot{\underline{\delta P}} \cdot \underline{\Omega} - \underline{\epsilon} \cdot \underline{\Omega} \times \underline{P} \right] \underline{P} + \underline{\delta P} \times \underline{P} \qquad (48)$$

In formulating the expression for the error in the estimate we shall use Equations 47 and 48 with d equal to zero. In the principle application considered here, the declination rarely exceeds twenty degrees and therefore is negligible in the expression for the error in the estimate of the error $\underline{\chi}$.

The error in estimate of $\underline{\chi}$ may be found by subtracting

Equation 47 from Equation 48 to form

$$\underline{\Delta X} = \underline{X} - \hat{\underline{X}} = -\frac{1}{\Omega^2}\left[\dot{\underline{\Delta P}}\cdot\underline{\Omega} + \underline{\epsilon}\cdot\underline{\Omega} \times \underline{P}\right] \underline{P} - \underline{\Delta P} \times \underline{P} \tag{49}$$

It is interesting to note the relation of the first term of Equation 49 to the inertial space derivatives of the same quantity. Specifically

$$\frac{d\underline{\underline{\Delta P}}}{dt} \cdot \underline{\Omega} = \frac{D\underline{\underline{\Delta P}}}{dt} \cdot \underline{\Omega} + \underline{\Omega} \times \underline{\Delta P}\cdot\underline{\Omega}$$

Since the last term is zero,

$$\frac{d\underline{\underline{\Delta P}}}{dt} \cdot \underline{\Omega} = \frac{D\underline{\underline{\Delta P}}}{dt} \cdot \underline{\Omega} = \dot{\underline{\Delta P}}\cdot\underline{\Omega} \tag{50}$$

It will be convenient to use the inertial space derivatives in the computation of errors which follow.

In summary, an estimate of $\underline{X}$, $\hat{\underline{X}}$ may be obtained from Equation 46; the error in the estimate given by Equation 49. In deriving these relations we have assumed that the inter-axis coupling is negligible during the period in which the computation and subsequent smoothing of data is to occur. Additionally, it has been assumed that the level errors, ζ_1 and ζ_2 about the $\underline{Y}_1$ and $\underline{Y}_2$ axes are small, and that $\dot{\zeta}_1$ and $\dot{\zeta}_2$ are negligible as a result.

C. Optimum Weighting Function

A device which implements Equation 46 provides three scaler outputs which are the components of $\underline{\chi}$, plus some unknown error $\underline{\Delta\chi}$, as a continuous function of time.[1] The ith component of $\underline{\chi}$ and $\underline{\Delta\chi}$ shall be denoted by χ_i and $\Delta\chi_i$ respectively. Data-smoothing techniques, usually referred to as "optimum filtering" in the continuous data case, may be applied to each of the scaler components to yield an optimum estimate of χ_i. The residual error, after filtering, shall be designated as $\delta\chi_i$. Specifically, we wish to apply a continuous linear operation to $\chi_i + \Delta\chi_i$ so that the ensemble average of the residual error squared, $\overline{\delta\chi_i^2}$, shall be minimized.

Inasmuch as the number of celestial bodies which can be tracked by radiometric means is very limited, the vector representing the direction to the star cannot be provided to the system throughout a 24-hour period. During the period when the star vector is not available, the system operates as a normal damped inertial navigator with the customary error propagation. Accordingly, there are two modes of operation - pure damped inertial alternating with periods of damped iner-

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[1]Normally, the computation would be performed by a digital computer whose output clearly is not a continuous function of time. However, if the computation is repeated at intervals of 1/2 B or less, where B is the equivalent star tracker bandwidth, the computer may be considered as a real time analogue device.

tial with celestial monitoring.

As a result of the foregoing discussion, one may describe the optimization problem as follows:

1. The system operates as a damped inertial navigator for extended periods until time t_0 when a star (sun) suitable for tracking is available.
2. At time t_0 a star vector is provided to the system. This vector provides a continuous estimate of the error vector $\underline{X}$. The initial value of $\underline{X}$, denoted by $\underline{X}_0$, is assumed to have components which are independent random variables with zero ensemble average.
3. A linear operation is applied to the components of $\underline{X} + \underline{\Delta X}$ for time $T = t - t_0$. The linear operator is to be chosen so that the ensemble average of the residual error squared is minimized at time $t_0 + T$.

It appears that there is no general analytic solution to the problem stated above. However it is possible to express the optimum weighting function (or linear operator) as an integral equation suitable for numerical solution by digital computer.

The output of a linear filter operating on the ith component of $\underline{X}$ may be expressed as

$$X_i + \delta X_i = \int_{t_0}^{t} W_i'(t, \tau)\left[X_i(\tau) + \Delta X_i(\tau)\right]d\tau \qquad (51)$$

where $W_i'(t, \tau)$ is a time-varying weighting function and δX_i is the residual error after filtering. Subtracting X_i from both sides of the equation gives the error

$$\delta\chi_1 = \int_{t_o}^{t} W_1'(t, \tau)\left[\chi_1(\tau) + \Delta\chi_1(\tau)\right]d\tau - \chi_1(t) \qquad (52)$$

By a change of variable Equation 52 may be written as

$$\delta\chi_1(t_o + T) = \int_0^T W_1(T, \tau)\left[\chi_1(t_o + \tau) + \Delta\chi_1(t_o + \tau)\right]d\tau$$
$$- \chi_1(t_o + T) \qquad (53)$$

where

$$W_1(T, \tau) = W_1'(t_o + T, t_o + \tau) \qquad (54)$$

In Equation 53, the variable t has been suppressed by making the substitution

$$t = t_o + T \qquad (55)$$

Squaring Equation 53 and taking the ensemble average[1] gives

$$\overline{\delta\chi_1^2} = \int_0^T\int_0^T W_1(T,\tau_1)W_1(T,\tau_2)\overline{\left[\chi_1(t_o + \tau_1) + \Delta\chi_1(t_o + \tau_1)\right] \times}$$
$$\overline{\left[\chi_1(t_o + \tau_2) + \Delta\chi_1(t_o + \tau_2)\right]}d\tau_1 d\tau_2 + \overline{\chi_1^2(t_o + T)} \qquad (56)$$
$$- 2\int_0^T W_1(T,\tau)\overline{\chi_1(t_o + T)\left[\chi_1(t_o + \tau) + \Delta\chi_1(t_o + \tau)\right]}d\tau$$

$W_1(t,\tau)$ is to be chosen so that $\overline{\delta\chi_1^2}$ given by Equation 56 is a

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[1]See Laning and Battin (2) or Davenport and Root (1) for a discussion of ensemble average as used here.

minimum. Applying the techniques of the calculus of variations, we replace $W_1(t,\tau)$ by $W_1(t,\tau) + aK(t,\tau)$ where $K(t,\tau)$ is an arbitrary function. If $W_1(t,\tau)$ produces a minimum in $\overline{\delta\chi_1^2}$, then the function $W_1(t,\tau) + aK(t,\tau)$ does not give a minimum unless a is equal to zero. As a result, $\overline{\delta\chi_1^2}$ is functionally related to the parameter a. Since $\overline{\delta\chi_1^2}$ is a minimum when a is zero, the derivative of $\overline{\delta\chi_1^2}(a)$ with respect to a must also be zero at a = 0. Thus, replacing $W_1(T,\tau)$ by $W_1(T,\tau) + aK(T,\tau)$ in Equation 56, taking the derivative with respect to a, and letting a approach zero gives

$$\left.\frac{d\overline{\delta\chi_1^2}}{da}\right|_{a=0} = \int_0^T\int_0^T \Big[W_1(T,\tau_1)K(T,\tau_2) + W_1(T,\tau_2)K(T,\tau_1)\Big] \overline{\left[\chi_1(t_o+\tau_1) + \Delta\chi_1(t_o+\tau_1)\right] \times \left[\chi_1(t_o+\tau_2) + \Delta\chi_1(t_o+\tau_2)\right]}\, d\tau_1 d\tau_2 - 2\int_0^T K(T,\tau)\overline{\chi_1(t_o+T)\left[\chi_1(t_o+\tau) + \Delta\chi_1(t_o+\tau)\right]}\, d\tau \tag{57}$$

Setting the derivative equal to zero, and noting the symmetry of the first term with respect to τ_1 and τ_2 gives

$$\int_0^T\int_0^T W_1(T,\tau_1)K(T,\tau_2)\overline{\left[\chi_1(t_o+\tau_1) + \Delta\chi_1(t_o+\tau_1)\right] \times \left[\chi_1(t_o+\tau_2) + \Delta\chi_1(t_o+\tau_2)\right]}\, d\tau_1 d\tau_2 - \int_0^T K(T,\tau)\overline{\chi_1(t_o+T)\left[\chi_1(t_o+\tau) + \Delta\chi_1(t_o+\tau)\right]}\, d\tau = 0 \tag{58}$$

Rearranging Equation 58 results in

$$\int_0^T K(T,\tau)d\tau \left\{ \int_0^T W_1(T,\tau_1)\overline{\left[\chi_1(t_o + \tau_1) + \Delta\chi_1(t_o + \tau_1) \right] \times} \right.$$

$$\overline{\left[\chi_1(t_o + \tau) + \Delta\chi_1(t_o + \tau) \right]} d\tau_1 \qquad (59)$$

$$\left. - \overline{\chi_1(t_o + T)\left[\chi_1(t_o + \tau) + \Delta\chi_1(t_o + \tau) \right]} \right\} = 0$$

Since Equation 59 must hold for arbitrary $K(t,\tau)$, the term in brackets must be zero. Thus

$$\int_0^T W_1(T,\tau_1)\overline{\left[\chi_1(t_o + \tau_1) + \Delta\chi_1(t_o + \tau_1) \right] \times}$$

$$\overline{\left[\chi_1(t_o + \tau) + \Delta\chi_1(t_o + \tau) \right]} d\tau_1 \qquad (60)$$

$$- \overline{\chi_1(t_o + T)\left[\chi_1(t_o + \tau) + \Delta\chi_1(t_o + \tau) \right]} = 0 \qquad 0 \leq \tau \leq T$$

Given the ensemble averages required, Equation 60 is the integral equation for the optimum time-varying weighting function $W_1(t,\tau)$. It should be noted that $W_1(t,\tau)$ differs from the weighting function to be implemented for the system, $W_1'(t,\tau)$. The relation between the two is repeated below for convenience:

$$W_1(t,\tau) = W_1'(t,\ t_o + \tau) \qquad (54)$$

or

$$W_1'(t,\tau) = W_1(t,\ \tau - t_o) \qquad (61)$$

For physical realizability, $W_1'(t,\tau) = 0$ if $\tau > t$ and $\tau < t_o$.

This implies that $W_i(T,\tau) = 0$ for $\tau > T$ as is to be expected.

Expanding the ensemble averages, it is seen that the following correlation functions are required:

$$c_1 = \overline{\chi_i(t_1)\chi_i(t_2)} \tag{62}$$

$$c_2 = \overline{\chi_i(t_2)\Delta\chi_i(t_1)} \tag{63}$$

$$c_3 = \overline{\chi_i(t_1)\Delta\chi_i(t_2)} \tag{64}$$

$$c_4 = \overline{\Delta\chi_i(t_1)\Delta\chi_i(t_2)} \tag{65}$$

1. <u>Description of error sources</u>

In deriving the expressions for the correlation functions c_1 through c_4 in terms of input parameters which follow, the ensemble averages of three quantities will be required: (1) $\overline{\chi_{i0}^2}$, (2) $\overline{\epsilon_i(t_1)\epsilon_i(t_2)}$, and (3) $\overline{\Delta P_i(t_1)\Delta P_i(t_2)}$. χ_{i0} is the initial value of the ith component of error χ, ϵ_i is the gyro drift rate of the gyro with input axis along z_i, and ΔP_i is the error in observation of the star vector along axis p_i.

Other cross correlation terms arise in the expressions for c_i, but because of the independence of ΔP, ϵ, and χ_0, these terms are zero. Specifically terms

$$\overline{\Delta P_i \chi_{j0}} = \overline{\Delta P_i \epsilon_j} = \overline{\chi_{i0}\epsilon_j} = 0 \tag{66}$$

From physical considerations, there is little doubt about the independence of $\underline{\Delta P}$ and $\underline{\epsilon}$, since they arise in two separate equipments. However, since the vector $\underline{\epsilon}$ is one of the driving functions for component χ_i, one may question the validity of setting $\overline{\chi_{i0}\epsilon_j}$ equal to zero. In fact, for the relatively short periods of time during which $\underline{\chi}$ is to be estimated (up to 4 hours), it is assumed in the derivations below that there is a specific short time functional dependence between χ_i and ϵ_i. On the other hand, many other sources of error contribute materially to $\underline{\chi}$ after extended periods of operation; errors in the accelerometers, gyros, and external velocity measurement of other channels are introduced into χ_i as a result of inter-axis coupling. These factors are ordinarily small, but after long operational periods it is reasonable to expect that the cumulative effect of all error sources make χ_{i0} essentially independent of ϵ_i. From similar arguments, it is assumed that

$$\overline{\chi_{i0}\chi_{j0}} = 0 \qquad j \neq i \tag{67}$$

Now consider the correlation $\overline{\epsilon_i\epsilon_j}$. Again from physical considerations we have that

$$\overline{\epsilon_i\epsilon_j} = 0 \quad \text{for} \quad i \neq j \tag{68}$$

since ϵ_i and ϵ_j arise in two different instruments. For i equal to j, we define

$$\overline{\epsilon_i(t_1)\epsilon_i(t_2)} = \phi_{\epsilon i}(\tau) \tag{69}$$

where $\tau = t_2 - t_1$. It is customary to assume that ϵ_i is a random process composed of two parts:

(1) A stationary random component, ϵ_i', with zero time and ensemble average;

(2) A random bias component ϵ_{i0} which represents the average value for the particular function $\epsilon_i(t)$ taken from the ensemble of possible functions. We assume that the ensemble average, $\overline{\epsilon_{i0}}$, is zero.

Accordingly, Equation 69 may be written as

$$\overline{\epsilon_i(t_1)\epsilon_i(t_2)} = \overline{[\epsilon_{i0} + \epsilon_i'(t_1)][\epsilon_{i0} + \epsilon_i'(t_2)]} \qquad (70)$$

Since $\epsilon_i'(t)$ is a stationary random variable with zero mean, Equation 70 becomes

$$\overline{\epsilon_i(t_1)\epsilon_i(t_2)} = \overline{\epsilon_{i0}^2} + \overline{\epsilon_i'(t_1)\epsilon_i'(t_2)} \qquad (71)$$

where the bar over the second term can represent either a time average or ensemble average. Further

$$\overline{\epsilon_i'(t_1)\epsilon_i'(t_2)} = \overline{\epsilon_i'(t_1 + \tau)\epsilon_i'(t_1)} \qquad (72)$$

The quantity on the right is simply the usual autocorrelation function for a stationary random process, denoted here by $\phi_{\epsilon i}'(\tau)$. Thus

$$\overline{\epsilon_i(t_1)\epsilon_i(t_2)} = \overline{\epsilon_{i0}^2} + \phi_{\epsilon i}'(\tau) \qquad (73)$$

where $\tau = t_2 - t_1$.

Finally consider the correlation of the components of $\underline{\Delta P}$.

In terms of the orthogonal unit vectors $\underline{p}_1$ and $\underline{p}_2$, $\underline{\Delta P}$ is

$$\underline{\Delta P} = \Delta P_1 \underline{p}_1 + \Delta P_2 \underline{p}_2$$

Since the tracking loops associated with the $\underline{p}_1$ and $\underline{p}_2$ axes are independent and since $\underline{p}_1$ and $\underline{p}_2$ are orthogonal,

$$\overline{\Delta P_1 \Delta P_2} = 0 \tag{74}$$

Further $\Delta P_i(t)$ is normally a stationary process so that

$$\overline{\Delta P_1(t_1)\Delta P_1(t_2)} = \overline{\Delta P_2(t_1)\Delta P_2(t_2)} = \phi_p(\tau) \tag{75}$$

where $\tau = t_2 - t_1$. In Equation 75 it is assumed that the tracking errors are statistically identical in each channel.

2. Derivation of the term $\overline{\chi_1(t_1)\chi_1(t_2)}$

Reference to Figure 5 shows that if the interaxis coupling is neglected, the short term characteristic of $\underline{\chi}$ may be expressed as

$$\chi_1 = \chi_{10} + \int_{t_0}^{t} \epsilon_1(\alpha)\,d\alpha \tag{76}$$

Thus using the notation defined in Equation 62, we may express $\overline{\chi_1(t_1)\chi_1(t_2)}$ as

$$c_1 = \overline{\chi_1(t_1)\chi_1(t_2)} = \overline{\left[\chi_{10} + \int_{t_0}^{t} \epsilon_1(\alpha_1)\,d\alpha_1\right] \times \left[\chi_{10} + \int_{t_0}^{t} \epsilon_1(\alpha_2)\,d\alpha_2\right]}$$

This may be written as

$$c_1 = \overline{\chi_{10}^2} + \int_{t_o}^{t}\int_{t_o}^{t} \overline{\epsilon_1(\alpha_1)\epsilon_1(\alpha_2)}d\alpha_1 d\alpha_2 \tag{77}$$

In Equation 77 we assume χ_{10} and ϵ_1 are independent as discussed in Paragraph III.C.1. above.

Letting $\alpha_1 = t_o + \beta_1$ and $\alpha_2 = t_o + \beta_2$ gives

$$c_1 = \overline{\chi_{10}^2} + \int_0^{t_2-t_o} d\beta_2 \int_0^{t_1-t_o} \overline{\epsilon_1(t_o + \beta_1)\epsilon_1(t_o + \beta_2)}d\beta_1 \tag{78}$$

Since ϵ_1 is a stationary random process, Equation 78 becomes

$$c_1 = \overline{\chi_{10}^2} + \int_0^{t_2-t_o} d\beta_2 \int_0^{t_1-t_o} \phi_{\epsilon 1}(\beta_1 - \beta_2)d\beta_1 \tag{79}$$

Substituting Equation 73 in Equation 79 gives

$$\begin{aligned} c_1 &= \overline{\chi_1(t_1)\chi_1(t_2)} \\ &= \overline{\chi_{10}^2} + \int_0^{t_2-t_o} d\beta_2 \int_0^{t_1-t_o} \left[\overline{\epsilon_{10}^2} + \phi'_{\epsilon 1}(\beta_1 - \beta_2) \right] d\beta_1 \end{aligned} \tag{80}$$

3. <u>Deviation of the term $\overline{\Delta\chi_1(t_1)\chi_1(t_2)}$</u>

The error $\underline{\Delta\chi}$ given by Equation 49 is

$$\underline{\Delta\chi} = -\frac{1}{\Omega^2}\left[\dot{\underline{\Delta P}}\cdot\underline{\Omega} + \underline{\epsilon}\cdot\underline{\Omega} \times \underline{P}\right] \underline{P} - \underline{\Delta P} \times \underline{P} \tag{81}$$

where $\dot{\underline{\Delta P}}$ represents differentiation with respect to time for any set of axes. Expressing $\underline{\Delta P}$ as

$$\underline{\Delta P} = \Delta P_1 \underline{p}_1 + \Delta P_2 \underline{p}_2 \tag{82}$$

and noting that Equation 81 is for zero declination gives

$$\underline{\Delta \chi} = \frac{1}{|\Omega|}\left[- \dot{\Delta P}_1 + \underline{\epsilon}\cdot\underline{p}_2 \right]\underline{P} + \Delta P_1 \underline{p}_2 - \Delta P_2 \underline{p}_1 \tag{83}$$

The ith component of $\underline{\Delta \chi}$ is obtained by forming the scaler product of $\underline{\Delta \chi}$ with $\underline{z}_i$

$$\Delta \chi_i = \left\{ \frac{1}{|\Omega|}\left[- \dot{\Delta P}_1 + \underline{\epsilon}\cdot\underline{p}_2 \right]\underline{P} + \underline{\Delta P}\cdot\underline{p}_2 - \Delta P_2 \underline{p}_1 \right\} \cdot \underline{z}_i \tag{84}$$

It is convenient to introduce the notation $Z_i^j(t)$ defined by

$$Z_i^j(t) = \underline{p}_j \cdot \underline{z}_i(t) \tag{85}$$

Using this notation with $\underline{P} = \underline{p}_3$, Equation 84 becomes

$$\Delta \chi_i(t) = \frac{1}{|\Omega|}\left[- \dot{\Delta P}_1(t) + \underline{\epsilon}(t)\cdot\underline{p}_2 \right] Z_i^3(t) + \Delta P_1 Z_i^2(t) - \Delta P_2 Z_i^1(t) \tag{86}$$

Multiplying Equation 76 by 86, and noting that χ_{i0} is not correlated with any term of Equation 86 gives

$$c_2 = \overline{\Delta \chi_i(t_1) \chi_i(t_2)} = \overline{\frac{1}{|\Omega|} Z_i^3(t_1) \underline{\epsilon}(t_1)\cdot p_2 \int_{t_0}^{t_2} \epsilon_i(\alpha)\, d\alpha} \tag{87}$$

In Equation 87, we have used the fact that $\underline{\epsilon}$ and $\underline{\Delta P}$ are uncorrelated.

The term $\underline{\epsilon}(t_1)\cdot\underline{p}_2$ may be written as

$$\sum_{u=1}^{3} \epsilon_u \underline{z}_u(t_1)\cdot\underline{p}_2 = \sum_{u=1}^{3} \epsilon_u(t_1) Z_u^2(t_1) \tag{88}$$

Thus

$$c_2 = \frac{1}{|\Omega|} \overline{\sum_{u=1}^{3} \epsilon_u(t_1) Z_u^2(t_1) Z_i^3(t_1) \int_{t_o}^{t_2} \epsilon_i(\alpha)\,d\alpha} \tag{89}$$

Interchanging the order of summation and integration gives, for c_2,

$$c_2 = \frac{1}{|\Omega|} \int_{t_o}^{t_2} d\alpha \sum_{u=1}^{3} \overline{\epsilon_u(t_1)\epsilon_i(\alpha)}\; Z_u^2(t_1) Z_i^3(t_1) \tag{90}$$

Since $\overline{\epsilon_u \epsilon_i} = 0$ for $u \neq i$, Equation 90 becomes

$$\begin{aligned} c_2 &= \frac{1}{|\Omega|} \int_{t_o}^{t_2} d\alpha\, \overline{\epsilon_i(t_1)\epsilon_i(\alpha)}\; Z_i^2(t_1) Z_i^3(t_1) \\ &= \frac{1}{|\Omega|} Z_i^2(t_1) Z_i^3(t_1) \int_{t_o}^{t_2} d\alpha \left[\overline{\epsilon_{io}^2} + \phi'_{\epsilon i}(\alpha - t_1) \right] \end{aligned} \tag{91}$$

Making the change of variable $\alpha = t_o + \beta$, Equation 91 may be written as

$$\begin{aligned} c_2 &= \overline{\Delta\chi_i(t_1)\chi_i(t_2)} \\ &= \frac{1}{|\Omega|} Z_i^2(t_1) Z_i^3(t_1) \int_0^{t_2 - t_o} \left[\overline{\epsilon_{io}^2} + \phi'_{\epsilon i}(\beta + t_o - t_1) \right] d\beta \end{aligned} \tag{92}$$

4. Derivation of the term $\overline{\chi_i(t_1)\Delta\chi_i(t_2)}$

The term $c_3 = \overline{\chi_i(t_1)\Delta\chi_i(t_2)}$ may be obtained by interchanging t_1 and t_2 in Equation 92. Thus

$$c_3 = \overline{\chi_i(t_1)\Delta\chi_i(t_2)}$$
$$= \frac{1}{|\Omega|} Z_i^2(t_2) Z_i^3(t_2) \int_0^{t_1 - t_o} \left[\overline{\epsilon_{io}^2} + \phi'_{\epsilon i}(\beta + t_o - t_2) \right] d\beta \tag{93}$$

5. Derivation of the term $\overline{\Delta\chi_i(t_1)\Delta\chi_i(t_2)}$

The equation for $\Delta\chi_i(t)$ is given in Equation 86.

Forming the product $\overline{\Delta\chi_i(t_1)\Delta\chi_i(t_2)} = c_4$ gives

$$c_4 = \frac{1}{\Omega^2} \overline{\left[\underline{\epsilon}(t_1)\cdot \underline{p}_2 - \Delta\dot{P}_1(t_1) \right] \times \left[\underline{\epsilon}(t_2)\cdot \underline{p}_2 - \Delta\dot{P}_1(t_2) \right]} Z_i^3(t_1) Z_i^3(t_2) \qquad \text{(a)}$$
$$+ \overline{\Delta P_1(t_1)\Delta P_1(t_2)}\; Z_i^2(t_1) Z_i^2(t_2) \qquad \text{(b)}$$
$$+ \overline{\Delta P_2(t_1)\Delta P_2(t_2)}\; Z_i^1(t_1) Z_i^2(t_2) \qquad \text{(c)} \qquad (94)$$
$$- \frac{1}{|\Omega|} \overline{\Delta\dot{P}_1(t_2)\Delta P_1(t_1)}\; Z_i^3(t_2) Z_i^2(t_1) \qquad \text{(d)}$$
$$- \frac{1}{|\Omega|} \overline{\Delta\dot{P}_1(t_1)\Delta P_1(t_2)}\; Z_i^3(t_1) Z_i^2(t_2) \qquad \text{(e)}$$

+ other cross product terms equal to zero since $\underline{\epsilon}$, ΔP_1, and ΔP_2 are all uncorrelated.

Consider term 94a. Since $\underline{\epsilon}$ and $\underline{\Delta P}$ are uncorrelated, it becomes

$$94a = \frac{1}{\Omega^2}\left\{\overline{\left[\underline{\epsilon}(t_1)\cdot\underline{p}_2\right]\left[\underline{\epsilon}(t_2)\cdot\underline{p}_2\right]} + \overline{\Delta\dot{P}_1(t_1)\Delta\dot{P}_1(t_2)}\right\} Z_1^3(t_1)Z_1^3(t_2) \tag{95}$$

Expansion of the terms in ϵ gives

$$94a = \frac{1}{\Omega^2}\left[\sum_{u=1}^{3}\sum_{v=1}^{3} \overline{\epsilon_u(t_1)\epsilon_v(t_2)}\; Z_u^2(t_1)Z_v^2(t_2)\right] Z_1^3(t_1)Z_1^3(t_2) + \overline{\Delta\dot{P}_1(t_1)\Delta\dot{P}_1(t_2)}\; Z_1^3(t_1)Z_1^3(t_2) \tag{96}$$

The gyro drift rates are uncorrelated so that Equation 96 becomes

$$94a = \frac{1}{\Omega^2}Z_1^3(t_1)Z_1^3(t_2)\left\{\sum_{u=1}^{3}\left[\overline{\epsilon_{u0}^2} + \phi'_{\epsilon u}(t_2 - t_1)\right]Z_u^2(t_1)Z_u^2(t_2) + \overline{\Delta\dot{P}_1(t_1)\Delta\dot{P}_1(t_2)}\right\} \tag{97}$$

Further, if the gyros are identical, then their statistical characteristics are identical. Under these circumstances Equation 97 is

$$94a = \frac{1}{\Omega^2}Z_1^3(t_1)Z_1^3(t_2)\left\{3\left[\overline{\epsilon_0^2} + \phi'_{\epsilon}(t_2 - t_1)\right]\sum_{u=1}^{3} Z_u^3(t_1)Z_u^2(t_2) + \overline{\Delta\dot{P}_1(t_1)\Delta\dot{P}_1(t_2)}\right\} \tag{98}$$

ΔP_1 is stationary so that the last term of Equation 98 may be expressed as a time average in the variable $\tau = t_2 - t_1$. Thus

$$\overline{\dot{\Delta P}_1(t_1)\dot{\Delta P}_2(t_2)} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \frac{d\left[\Delta P_1(t+\tau)\right]}{dt} \frac{d\left[\Delta P_1(t)\right]}{dt} dt \tag{99}$$

Integrating Equation 99 by parts gives

$$\overline{\dot{\Delta P}_1(t_1)\dot{\Delta P}_1(t_2)} = \lim_{T \to \infty} \frac{1}{2T}\left[\frac{d\Delta P(t+\tau)}{dt} \Delta P_1(t) \Big|_{-T}^{T} - \int_{-T}^{T} \Delta P_1(t) \frac{d^2 \Delta P_1(t+\tau)}{dt^2} dt \right] \tag{100}$$

For physical processes the first term is zero. Further the derivative with respect to time in the second term may be replaced by the derivative with respect to τ since the variable in the integrand is $t + \tau$. If we change the order of integration, limit, and differentiation, Equation 100 becomes

$$\overline{\dot{\Delta P}_1(t_1)\dot{\Delta P}_1(t_2)} = -\frac{d^2}{d\tau^2} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \Delta P_1(t)\Delta P_1(t+\tau)dt \tag{101}$$

The term under the integral is simply the autocorrelation function of ΔP_1. Hence

$$\overline{\dot{\Delta P}_1(t_1)\dot{\Delta P}_1(t_2)} = -\frac{d^2}{d\tau^2} \phi_p(\tau)\Big|_{\tau = t_2 - t_1} \tag{102}$$

where $\tau = t_2 - t_1$.

Thus the term 94a is given by

$$94a = \frac{1}{\Omega^2} Z_i^3(t_1) Z_i^3(t_2) \left\{ - \frac{d^2}{d\tau^2} \phi_p(\tau) \bigg|_{\tau = t_2 - t_1} + 3\left[\overline{\epsilon_0^2} + \phi_\epsilon'(t_2 - t_1)\right] \sum_{u=1}^{3} Z_u^3(t_1) Z_u^2(t_2) \right\} \tag{103}$$

The term 94b is immediately recognized as the autocorrelation function of ΔP_1 multiplied by the geometric factors Z_i^j. Thus

$$94b = \phi_p(t_2 - t_1) Z_i^2(t_1) Z_i^2(t_2) \tag{104}$$

Similarly

$$94c = \phi_p(t_2 - t_1) Z_i^1(t_1) Z_i^1(t_2) \tag{105}$$

Using a process similar to that used in Equation 102, we may compute the term 94d as follows

$$\begin{aligned} 94d &= - \frac{1}{|\Omega|} Z_i^3(t_2) Z_i^2(t_1) \overline{\Delta\dot{P}_1(t_2)\Delta P_1(t_1)} \\ &= - \frac{1}{|\Omega|} Z_i^3(t_2) Z_i^2(t_1) \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \frac{d\Delta P_1(t + \tau)}{dt} \Delta P_1(t) dt \end{aligned} \tag{106}$$

Replacing the derivative with respect to t by one with respect to τ, and interchanging the order of operations gives

$$94d = -\frac{1}{|\Omega|} Z_1^3(t_2) Z_1^2(t_1) \frac{d}{d\tau} \phi_p(\tau) \Big|_{\tau = t_2 - t_1} \tag{107}$$

where $\tau = t_2 - t_1$.

The term 94e may be written immediately from Equation 107 by interchanging t_1 and t_2, and noting that[1]

$$\frac{d}{d\tau} \phi_p(\tau) \Big|_{\tau=\tau_1} = -\frac{d}{d\tau} \phi_p(\tau) \Big|_{\tau=-\tau_1}$$

Thus

$$94e = +\frac{1}{|\Omega|} Z_1^3(t_1) Z_1^2(t_2) \frac{d}{d\tau} \phi_p(\tau) \Big|_{\tau = t_2 - t_1} \tag{108}$$

Adding Equations 95, 103, 104, 105, 107, and 108 give, for c_4,

$$\begin{aligned} c_4 &= \overline{\Delta\chi_1(t_1)\Delta\chi_1(t_2)} \\ &= \frac{1}{\Omega^2} Z_1^3(t_1) Z_1^3(t_2) \Big\{ -\frac{d^2}{d\tau^2} \phi_p(\tau) \Big|_{\tau = t_2 - t_1} \\ &\quad + 3\Big[\overline{\epsilon_0^2} + \phi_\epsilon'(\tau) \sum_{u=1}^{3} Z_u^3(t_1) Z_u^2(t_2)\Big\} \\ &\quad + \phi_p(t_2 - t_1)\Big[Z_1^2(t_1) Z_1^2(t_2) + Z_1^1(t_1) Z_1^1(t_2)\Big] \\ &\quad - \frac{1}{|\Omega|} \frac{d}{d\tau} \phi_p(\tau) \Big|_{\tau = t_2 - t_1} \Big[Z_1^3(t_2) Z_1^2(t_1) - Z_1^3(t_1) Z_1^2(t_2)\Big] \end{aligned} \tag{109}$$

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[1]See Laning and Battin (2).

For convenience we repeat Equations 80, 92, 93, and 60 below

$$c_1 = \overline{\chi_1(t_1)\chi_1(t_2)}$$
$$= \overline{\chi_{10}^2} + \int_0^{t_2-t_o} d\beta_2 \int_0^{t_1-t_o} \left[\overline{\epsilon_0^2} + \phi_\epsilon'(\beta_1 - \beta_2) \right] d\beta_1 \tag{80}$$

$$c_2 = \overline{\Delta\chi_1(t_1)\chi_1(t_2)}$$
$$= \frac{1}{|\Omega|} Z_1^2(t_1) Z_1^3(t_1) \int_0^{t_2-t_o} \left[\overline{\epsilon_0^2} + \phi_\epsilon'(\beta + t_o - t_1) \right] d\beta \tag{92}$$

$$c_3 = \overline{\chi_1(t_1)\Delta\chi_1(t_2)}$$
$$= \frac{1}{|\Omega|} Z_1^2(t_2) Z_1^3(t_2) \int_0^{t_1-t_o} \left[\overline{\epsilon_0^2} + \phi_\epsilon'(\beta + t_o - t_2) \right] d\beta \tag{93}$$

The integral equation for the optimum weighting function is

$$\int_0^T W_1(T,\tau_1) \overline{\left[\chi_1(t_o + \tau_1) + \Delta\chi_1(t_o + \tau_1) \right] x}$$
$$\overline{\left[\chi_1(t_o + \tau) + \Delta\chi_1(t_o + \tau) \right]} d\tau_1 \tag{60}$$
$$- \overline{\chi_1(t_o + T) \left[\chi_1(t_o + \tau) + \Delta\chi_1(t_o + \tau) \right]} = 0$$

Each of the terms required in Equation 60 are given above in Equations 80, 92, 93, and 109. In order to see the essential characteristic of Equation 60, it may be written in the following form:

$$\int_0^T W_i(T,\tau_1)J_i(\tau,\tau_1)d\tau_1 - I_i(T,\tau) \tag{110}$$

In Equation 110, $J_i(\tau,\tau_1)$ is

$$J_i(\tau,\tau_1) = \overline{X_{i0}^2} + \int_0^\tau d\beta_2 \int_0^{\tau_1} [\overline{\epsilon_0^2} + \phi_\epsilon'(\beta_1 - \beta_2)]d\beta_1$$

$$+ \frac{1}{|\Omega|} Z_i^2(t_o + \tau)Z_i^3(t_o + \tau)\int_0^{\tau_1}[\overline{\epsilon_0^2} + \phi_\epsilon'(\beta - \tau)]d\beta$$

$$+ \frac{1}{|\Omega|} Z_i^2(t_o + \tau_1)Z_i^3(t_o + \tau_1)\int_0^{\tau}[\overline{\epsilon_0^2} + \phi_\epsilon'(\beta - \tau_1)]d\beta$$

$$+ \frac{1}{\Omega^2} Z_i^3(t_o + \tau_1)Z_i^3(t_o + \tau)\left\{- \frac{d^2}{d\beta^2}\phi_p(\beta)\Big|_{\beta = \tau-\tau_1}\right.$$

$$\left. + 3[\overline{\epsilon_0^2} + \phi_\epsilon'(\tau - \tau_1)]\sum_{u=1}^{3} Z_u^3(t_o + \tau_1)Z_u^2(t_o + \tau)\right\}$$

$$+ \phi_p(\tau-\tau_1)[Z_i^2(t_o+\tau_1)Z_i^2(t_o+\tau) + Z_i^1(t_o+\tau_1)Z_i^1(t_o+\tau)]$$

$$- \frac{1}{|\Omega|}\frac{d}{d\beta}\phi_p(\beta)\Big|_{\beta=\tau-\tau_1}[Z_i^3(t_o+\tau)Z_i^2(t_o+\tau_1)$$

$$- Z_i^3(t_o+\tau_1)Z_i^2(t_o+\tau)] \tag{111}$$

Similarly I_i may be expressed as

$$I_i(T,\tau) = \overline{X_{i0}^2} + \int_0^\tau d\beta_2 \int_0^T [\overline{\epsilon_0^2} + \phi_\epsilon'(\beta_1 - \beta_2)]d\beta_1$$

$$+ \frac{1}{|\Omega|} Z_i^2(t_o + \tau)Z_i^3(t_o + \tau)\int_0^T[\overline{\epsilon_0^2} + \phi_\epsilon'(\beta - \tau)]d\beta \tag{112}$$

Unless the forms of ϕ_ϵ' and ϕ_p are specified, Equations 111 and

112 cannot be reduced further.

D. Residual Error after Optimum Filtering

The ensemble average error squared is given by Equation 56. Rearranging gives

$$\overline{\delta X_1^2(t_o + T)} = \overline{X_1^2(t_o + T)} + \int_0^T W_1(T,\tau)d\tau \left\{ \int_0^T W_1(T,\tau_1) \overline{[X_1(t_o+\tau_1)+\Delta X_1(t_o+\tau_1)] \times [X_1(t_o + \tau) + \Delta X_1(t_o + \tau)]} d\tau_1 - 2\, \overline{X_1(t_o + T)[X_1(t_o + \tau) + \Delta X_1(t_o + \tau)]} \right\} \tag{113}$$

Substituting Equation 60 for the integral with respect to τ_1 gives the error after optimal filtering as

$$\overline{\delta X_1^2} = \overline{X_1^2(t_o + T)} - \int_0^T W_1(T,\tau)\, \overline{X_1(t_o + T)[X_1(t_o + \tau) + \Delta X_1(t_o + \tau)]} d\tau \tag{114}$$

It should be noted that the ensemble average in the integrand of the second term is given explicitly by Equation 112. The first term may be obtained from Equation 80, and is

$$\overline{X_1^2(t_o + T)} = \overline{X_{10}^2} + \int_0^T d\beta_2 \int_0^T [\overline{\epsilon_0^2} + \phi_\epsilon'(\beta_1 - \beta_2)]d\beta_1 \tag{115}$$

Equation 114 gives the residual error of the estimate of

X_i at time $t_o + T$ after optimum filtering has been performed on the raw estimate.

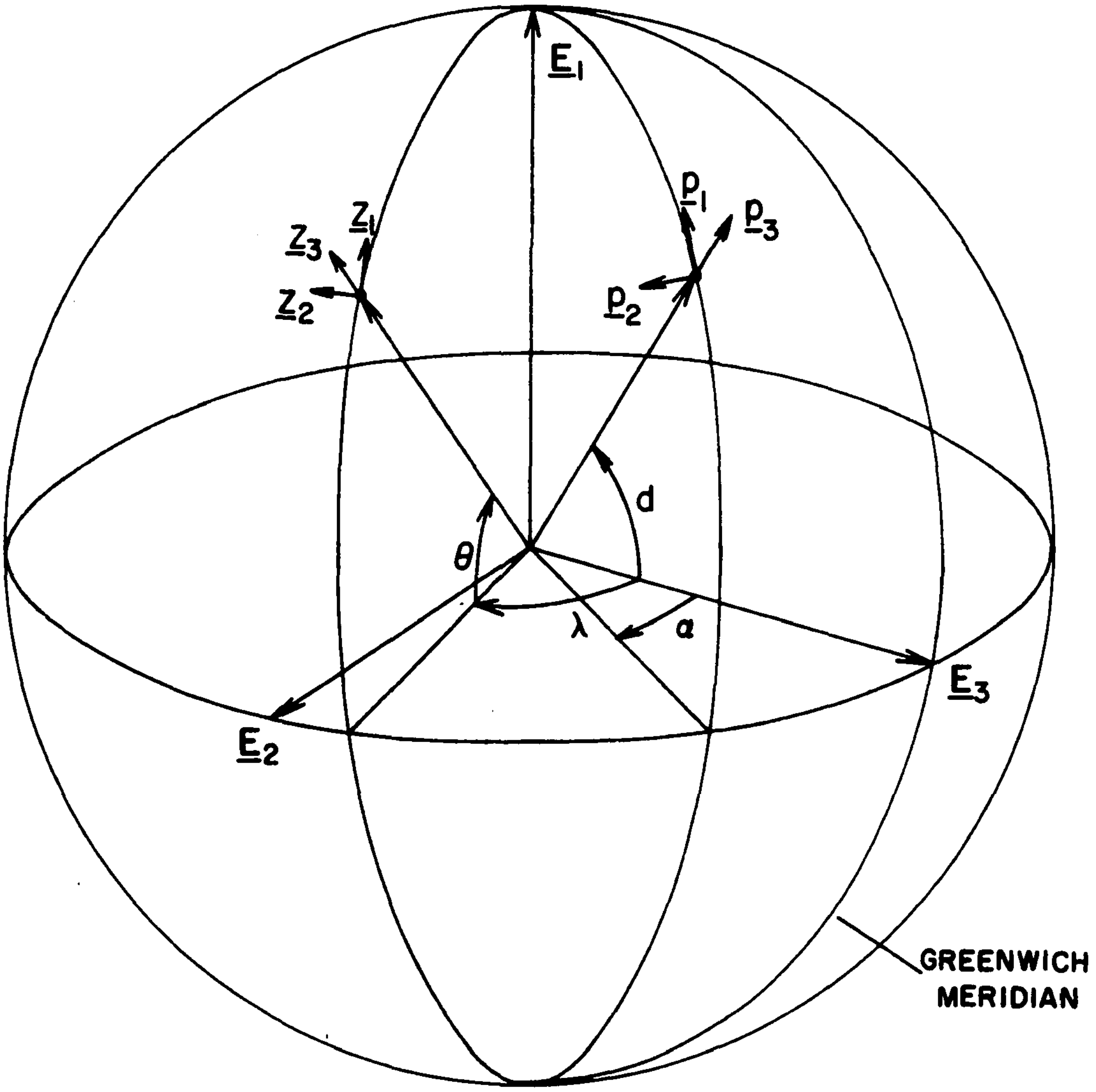
E. Computation of the Geometric Factors $Z_i^j(t)$

In Section III.C. the factor $Z_i^j(t)$ was defined as

$$Z_i^j(t) = \underline{p}_j \cdot \underline{z}_i(t) \tag{116}$$

where $\underline{p}_j$ and $\underline{z}_i$ are the unit vectors of the p- and z-coordinate systems respectively. The position of the z-coordinate system is specified in terms of the coordinates latitude θ and longitude λ relative to a coordinate system fixed with respect to the earth, i.e., a coordinate system with one axis colinear with the polar axis, one axis in the plane of the equator and passing through the Greenwich meridian, and a third orthogonal axis which forms a right-hand set. This coordinate system shall be designated as the E-coordinate system, with unit vectors $\underline{E}_1$, $\underline{E}_2$, and $\underline{E}_3$. Similarly, the p-coordinate system is customarily specified in terms of the declination, d, and Greenwich hour angle, α, relative to the E-coordinate system. It should be noted that both the p- and z-coordinate systems have one axis north. Figure 7 illustrates the relative orientations.

Applying the laws of spherical trigonometry, one may form the following arrays:



α = GREENWICH HOUR ANGLE — WEST

λ = LONGITUDE — WEST

θ = LATITUDE

d = DECLINATION

FIGURE 7. GEOMETRY FOR COMPUTATION OF FACTORS Z_i^j.

	z_1	z_2	z_3
E_1	$\cos\theta$	0	$\sin\theta$
E_2	$-\sin\lambda\sin\theta$	$\cos\lambda$	$\sin\lambda\cos\theta$
E_3	$-\cos\lambda\sin\theta$	$-\sin\lambda$	$\cos\lambda\cos\theta$

	p_1	p_2	p_3
E_1	$\cos d$	0	$\sin d$
E_2	$-\sin\alpha\sin d$	$\cos\alpha$	$\sin\alpha\cos d$
E_3	$-\cos\alpha\sin d$	$-\sin\alpha$	$\cos\alpha\cos d$

If the first array is defined as the matrix A with general term a_{ij}, the term a_{ij} is the cosine of the angle between the unit vector $\underline{E}_i$ and the unit vector $\underline{z}_j$. Similarly, defining the second array as the matrix B with general term b_{ij}, the term b_{ij} is the cosine of the angle between the $\underline{E}_i$ axis and the $\underline{p}_j$ axis.

Now consider an arbitrary column vector, R_E, with components expressed in the E-system. The components of R expressed in the z-system, R_z, are given by

$$R_z = A'R_E \tag{117}$$

where A' is the transpose of the matrix A. Similarly, R expressed in the p-coordinate system, R_p, is given by

$$R_p = B'R_E \tag{118}$$

Solving Equation 117 for R_E and substituting in Equation 118 gives

$$R_p = B'AR_z \tag{119}$$

In Equation 119, use has been made of the fact that A and B are orthogonal matrices; thus the inverse is equal to the transpose.

The product A'B, designated as the matrix C is

$$C = \tag{120}$$

$$\begin{pmatrix} \cos d \cos \theta + \sin d \sin \theta \cos(\lambda-\alpha) & -\sin \theta \sin(\lambda-\alpha) & \cos \theta \sin d - \sin \theta \cos d \cos(\lambda-\alpha) \\ \sin d \sin(\lambda-\alpha) & \cos(\lambda-\alpha) & -\cos d \sin(\lambda-\alpha) \\ \sin \theta \cos d - \cos \theta \sin d \cos(\lambda-\alpha) & \cos \theta \sin(\lambda-\alpha) & \sin \theta \sin d + \cos d \cos \theta \cos(\lambda-\alpha) \end{pmatrix}$$

Inspection of Equation 119 reveals that general term of the matrix product A'B, denoted here as c_{ij}, is simply the cosine of the angle between the unit vectors $\underline{p}_j$ and $\underline{z}_i$. As an example, the cosine of the angle between the $\underline{z}_1$ and $\underline{p}_2$ axes is

$$- \sin \theta \sin (\lambda-\alpha)$$

From Equation 116, it is seen that the general term c_{ij} is simply z_i^j. Thus

$$z_i^j = c_{ij} \tag{121}$$

For computation of the optimum weighting function described in this section, the factors Z_i^j for zero declination are needed. For this case the matrix C becomes

$$C = \begin{pmatrix} \cos\theta & -\sin\theta\sin(\lambda-\alpha) & -\sin\theta\cos(\lambda-\alpha) \\ 0 & \cos(\lambda-\alpha) & -\sin(\lambda-\alpha) \\ \sin\theta & \cos\theta\sin(\lambda-\alpha) & \cos\theta\cos(\lambda-\alpha) \end{pmatrix} \quad (122)$$

Accordingly, if the declination is zero, the factors Z_i^j may be tabulated as shown below:

$$Z_1^1(t) = \cos\theta \quad (123)$$

$$Z_1^2(t) = -\sin\theta\sin[\lambda-\alpha(t)] \quad (124)$$

$$Z_1^3(t) = -\sin\theta\cos[\lambda-\alpha(t)] \quad (125)$$

$$Z_2^1(t) = 0 \quad (126)$$

$$Z_2^2(t) = \cos[\lambda-\alpha(t)] \quad (127)$$

$$Z_2^3(t) = -\sin[\lambda-\alpha(t)] \quad (128)$$

$$Z_3^1(t) = \sin\theta \quad (129)$$

$$Z_3^2(t) = \cos\theta\sin[\lambda-\alpha(t)] \quad (130)$$

$$Z_3^3(t) = \cos\theta\cos[\lambda-\alpha(t)] \quad (131)$$

It should be noted that the latitude, θ, and the longitude, λ, have been treated as quasi-constants. The quantity $\lambda-\alpha(t)$ is usually referred to as the local hour angle of the star.

IV. RESTRICTED SOLUTION FOR $W_1(T,\tau)$

In the general case, there is no closed form analytic solution for the optimum weighting function $W_i(T,\tau)$ given by the integral Equation 60. Accordingly, the design of a hybrid celestial-inertial navigation system using the techniques described in Section III must of necessity be based on numerical solution of the integral equation. However, it is informative to consider two special cases for which solutions may be obtained in order to gain some insight into the nature of the problem. In both cases an equatorial geometry shall be assumed, i.e., zero latitude and declination.

A. Solution for $W_1(T,\tau)$ with T Approaching Zero

Equation 110 gives the integral equation for $W_1(T,\tau)$ as

$$\int_0^T W_1(T,\tau_1)J_1(\tau,\tau_1)d\tau_1 - I_1(T,\tau) = 0 \tag{132}$$

The functions $J_1(\tau,\tau_1)$ and $I_1(T,\tau)$ may be obtained from Equations 111 and 112 with $i = 1$. For the case considered here, they are

$$J_1(\tau,\tau_1) = \overline{x_{10}^2} + \int_0^\tau d\beta_2\int_0^{\tau_1}\left[\overline{\epsilon_0^2} + \phi_\epsilon'(\beta_1-\beta_2)\right]d\beta_1 + \phi_p(\tau-\tau_1) \tag{133}$$

and

$$I_1(T,\tau) = \overline{x_{10}^2} + \int_0^\tau d\beta_2\int_0^T\left[\overline{\epsilon_0^2} + \phi_\epsilon'(\beta_1-\beta_2)\right]d\beta_1 \tag{134}$$

In Equations 133 and 134, Equations 123 through 131, with $\theta = 0$, have been substituted for the factors $Z_i^j(t)$ appearing in Equations 111 and 112.

For typical gyroscopes and star trackers, the functions $\phi_\epsilon'(\tau)$ and $\phi_p(\tau)$ are of the form

$$\phi_\epsilon'(\tau) = a^2 e^{-c|\tau|}$$

and

$$\phi_p(\tau) = b^2 e^{-d|\tau|} \tag{135}$$

Since $0 \leq \tau,\ \tau_1 \leq T$, for T approaching zero, Equations 133 and 134 reduce to

$$J_1(\tau,\tau_1) \cong \overline{\chi_{10}^2} + b^2 \tag{136}$$

and

$$I_1(T,\tau) = \overline{\chi_{10}^2} \tag{137}$$

If Equations 136 and 137 are substituted in Equation 132, the integral equation for $W_1(T,\tau)$ becomes

$$\lim_{T \to 0} \int_0^T W_1(T,\tau_1)\left[\overline{\chi_{10}^2} + b^2\right]d\tau_1 = \overline{\chi_{10}^2} \tag{138}$$

This equation is valid only if

$$W_1(T,\tau) = k\delta(\tau) \tag{139}$$

where

$$k = \frac{\overline{\chi_{10}^2}}{\overline{\chi_{10}^2} + b^2} \tag{140}$$

The unit impulse function at $\tau = 0$ is denoted by $\delta(\tau)$.

Equations 139 and 140 place in evidence the dependence of the optimum weighting function on the relative magnitudes of the parameters $\overline{\chi_{10}^2}$ and b^2 (and in the general case, $\overline{\epsilon_0^2}$ as well). This is due to the fact that in deriving the integral equation for $W_1(T,\tau)$ we did not require a perfect solution for the error vector $\underline{\chi}$ in the absence of instrumentation errors. Clearly, if the parameter $\overline{\chi_{10}^2}$ is very small relative to $\phi_\epsilon(0)$ and $\phi_p(0)$,[1] one would not want to heavily weight the information provided by the star vector in forming the estimate of the vector $\underline{\chi}$. This fact is reflected by Equation 140 which shows that k is very small under these circumstances. Conversely, if $\overline{\chi_{10}^2}$ is large relative to $\phi_\epsilon(0)$ and $\phi_p(0)$, the star vector information should be strongly weighted. In this case k becomes unity.

The ensemble average of the residual error squared may be obtained from Equations 114 and 115. For T approaching zero, Equation 115 becomes

$$\overline{\chi_1^2(t_o)} = \overline{\chi_{10}^2} \tag{141}$$

Noting that the correlation function in the integrand of Equation 114 is simply $I_1(T,\tau)$ where $I_1(T,\tau)$ is given by

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[1] $\phi_\epsilon(0)$ and $\phi_p(0)$ give the mean square value of the gyro and star vector measurement errors respectively. See Laning and Battin (2).

Equation 137, the ensemble error squared given by Equation 114 becomes

$$\overline{\delta X_1^2} = \overline{X_1^2(t_o)} - \lim_{T \to 0} \int_0^T W_1(T,\tau) I_1(T,\tau) d\tau \qquad (142)$$

Substituting Equation 139 for $W_1(T,\tau)$ and Equation 137 for $I_1(T,\tau)$ gives

$$\overline{\delta X_1^2(t_o)} = \overline{X_1^2(t_o)} - \lim_{T \to 0} k\int_0^T \delta(\tau)\overline{X_{10}^2}\, d\tau \qquad (143)$$

Integration yields

$$\overline{\delta X_1^2(t_o)} = \frac{\overline{X_{10}^2}\, b^2}{\overline{X_{10}^2} + b^2} \qquad (144)$$

In obtaining Equation 144, it should be noted that $\overline{X_1^2(t_o)}$ is equal to $\overline{X_{10}^2}$.

Again let us examine the case for $\overline{X_{10}^2}$ large relative to b^2. The residual error is then simply

$$\overline{\delta X_1^2} \cong b^2 \qquad (145)$$

which is the mean square error in the measurement of the star vector. We recall that k given by Equation 140 is unity for this case. Since the solution has been restricted to small T, one would not expect the optimal filter to be effective in reducing the residual error. This fact is reflected by Equation 145.

If $\overline{x_{10}^2}$ is small relative to b^2, the error becomes

$$\overline{\delta x_1^2} \cong \overline{x_{10}^2} \tag{146}$$

The coefficient k in this case becomes very small indicating that the optimal filter has little effect on the residual error as is to be expected in view of Equation 146.

In assuming that T is very small, the errors due to gyro drift rate have not entered the problem. This restriction is removed in part in the following section.

B. Solution for $W_1(T,\tau)$ for Arbitrary T

Integral Equation 132 for $W_1(T,\tau)$ with correlation functions $J_1(\tau,\tau_1)$ and $I_1(T,\tau)$ given by Equations 133 and 134 respectively, apply to the case considered here. We now assume that

$$\phi_\epsilon'(\tau) = 0 \tag{147}$$

and

$$\phi_p(\tau) = b^2\delta(\tau) \tag{148}$$

where $\delta(\tau)$ is the unit impulse function.[1]

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[1]Inasmuch as autocorrelation functions are symmetric, the unit impulse function used here is symmetric, i.e. $\delta(t-t_0) = \delta(t_0-t)$. As a result of this definition we have

$$\int_0^T \delta(\tau)d\tau = \int_0^T \delta(T-\tau)d\tau = 1/2$$

These assumptions are not as restrictive as they may seem. In precision gyroscopes, the bias component of gyro drift rate is typically an order of magnitude larger than the random component. Further, the reciprocal of the effective noise (error) bandwidth of practical star trackers may be much smaller than the parameter T, and hence, Equation 148 is a valid approximation for $\phi_p(\tau)$.

Substitution of Equation 147 and 148 into Equations 133 and 134 give

$$J_1(\tau,\tau_1) = \overline{\chi_{10}^2} + \int_0^{\tau} d\beta_2 \int_0^{\tau_1} \overline{\epsilon_0^2} d\beta_1 + b^2\delta(\tau - \tau_1) \qquad (149)$$

and

$$I_1(T,\tau) = \overline{\chi_{10}^2} + \int_0^{\tau} d\beta_2 \int_0^{T} \epsilon_0^2 d\beta_1 \qquad (150)$$

Integration of Equations 149 and 150 yield

$$J_1(\tau,\tau_1) = \overline{\chi_{10}^2} + \overline{\epsilon_0^2}\tau\tau_1 + b^2\delta(\tau - \tau_1) \qquad (151)$$

and

$$I_1(T,\tau) = \overline{\chi_{10}^2} + \overline{\epsilon_0^2}\, T\tau \qquad (152)$$

Using Equations 151 and 152 in Equation 132 gives the integral equation for $W_1(T,\tau)$ as

$$\int_0^T W_1(T,\tau_1)\left[\overline{\chi_{10}^2} + \overline{\epsilon_0^2}\tau\tau_1 + b^2\delta(\tau-\tau_1)\right]d\tau_1 = \overline{\chi_{10}^2} + \overline{\epsilon_0^2}T\tau \qquad (153)$$

Let us assume a solution of the form

$$W_1(T,\tau_1) = \begin{bmatrix} C_1 + C_2\,\tau_1; & 0 < \tau_1 < T \\ 2C_3 \quad ; & \tau_1 = 0 \\ 2C_4 \quad ; & \tau_1 = T \end{bmatrix} \tag{154}$$

Substituting this trial solution in Equation 153 gives

$$\int_0^T \begin{bmatrix} C_1 + C_2\tau_1 \\ 2C_3 \\ 2C_4 \end{bmatrix} \left[\overline{\chi_{10}^2} + \overline{\epsilon_0^2}\tau\tau_1 \right] d\tau_1$$

$$+ \int_0^T \begin{bmatrix} C_1 + C_2\tau_1 \\ 2C_3 \\ 2C_4 \end{bmatrix} b^2\delta(\tau-\tau_1)\,d\tau_1 = \overline{\chi_{10}^2} + \overline{\epsilon_0^2}\tau T \tag{155}$$

The value of the integrand in the first integral may be arbitrarily chosen at a finite number of points. Accordingly we choose $W_1(T,0) = C_1$ and $W_1(T,T) = C_1 + C_2T$. As a result, the following equation is valid for all $0 \leq \tau_1 \leq T$:

$$\int_0^T \begin{bmatrix} C_1 + C_2\tau_1 \\ 2C_3 \\ 2C_4 \end{bmatrix} \left[\overline{\chi_{10}^2} + \overline{\epsilon_0^2}\tau\tau_1 \right] d\tau_1$$

$$= \int_0^T (C_1 + C_2\tau_1)\left[\overline{\chi_{10}^2} + \overline{\epsilon_0^2}\tau\tau_1 \right] d\tau_1 \tag{156}$$

Integration of Equation 156 gives

$$\int_0^T \begin{bmatrix} C_1 + C_2\tau_1 \\ 2C_3 \\ 2C_4 \end{bmatrix} \left[\overline{\chi_{10}^2} + \overline{\epsilon_0^2}\tau\tau_1 \right] d\tau_1 \tag{157}$$

$$= C_1\left[\overline{\chi_{10}^2}T + \frac{\overline{\epsilon_0^2}\tau T^2}{2} \right] + C_2\left[\frac{\overline{\chi_{10}^2}T^2}{2} + \frac{\overline{\epsilon_0^2}\tau T^3}{3} \right]$$

The second integral of Equation 155 is

$$\int_0^T \begin{bmatrix} C_1 + C_2\tau_1 \\ 2C_3 \\ 2C_4 \end{bmatrix} b^2\delta(\tau-\tau_1)d\tau_1 = \begin{bmatrix} b^2(C_1 + C_2\tau); & 0 < \tau < T \\ b^2C_3 \quad ; & \tau = 0 \\ b^2C_4 \quad ; & \tau = T \end{bmatrix} \tag{158}$$

Substituting Equations 157 and 158 into 155 give the following set of equations:

For $0 < \tau < T$,

$$C_1\left[\overline{\chi_{10}^2}T + \frac{\overline{\epsilon_0^2}\tau T^2}{2} \right] + C_2\left[\frac{\overline{\chi_{10}^2}T^2}{2} + \frac{\overline{\epsilon_0^2}\tau T^3}{3} \right]$$

$$+ b^2(C_1 + C_2\tau) = \overline{\chi_{10}^2} + \overline{\epsilon_0^2}\tau T \tag{159}$$

For $\tau = 0$,

$$C_1\overline{\chi_{10}^2}T + \frac{C_2\overline{\chi_{10}^2}T^2}{2} + b^2C_3 = \overline{\chi_{10}^2} \tag{160}$$

For $\tau = T$,

$$C_1T\left[\overline{x_{10}^2} + \frac{\overline{\epsilon_0^2}T^2}{2}\right] + C_2T^2\left[\frac{\overline{x_{10}^2}}{2} + \frac{\overline{\epsilon_0^2}T^2}{3}\right] + b^2C_4 = \overline{x_{10}^2} + \overline{\epsilon_0^2}T^2 \tag{161}$$

Equation 159 must hold for all $0 < \tau < T$; therefore, equating the constant terms and the τ dependent terms gives the following pair of simultaneous equations in C_1 and C_2:

$$C_1[\overline{x_{10}^2}T + b^2] + C_2\frac{\overline{x_{10}^2}T^2}{2} = \overline{x_{10}^2} \tag{162}$$

and

$$C_1\frac{\overline{\epsilon_0^2}T^2}{2} + C_2\left[\frac{\overline{\epsilon_0^2}T^3}{3} + b^2\right] = \overline{\epsilon_0^2}T \tag{163}$$

Solution of Equations 162 and 163 for C_1 and C_2 yield

$$C_1 = \frac{2\overline{x_{10}^2}[6b^2 - \overline{\epsilon_0^2}T^3]}{\overline{x_{10}^2}T[\overline{\epsilon_0^2}T^3 + 12b^2] + 4b^2[\overline{\epsilon_0^2}T^3 + 3b^2]} \tag{164}$$

and

$$C_2 = \frac{6\overline{\epsilon_0^2}T[\overline{x_{10}^2}T + 2b^2]}{\overline{x_{10}^2}T[\overline{\epsilon_0^2}T^3 + 12b^2] + 4b^2[\overline{\epsilon_0^2}T^3 + 3b^2]} \tag{165}$$

The constants C_3 and C_4 may be obtained from the pair of Equations 160 and 161. Solution gives

$$C_3 = \frac{C_1}{\overline{x_{10}^2}} \tag{166}$$

and

$$C_4 = \frac{C_1}{\overline{x_{10}^2}}\left[1 + \frac{\overline{\epsilon_0^2}T^2}{3\overline{x_{10}^2}}\right] + \frac{\overline{\epsilon_0^2}T^2}{6b^2}(4 - C_2T^2) \tag{167}$$

Thus Equation 154, together with Equations 164, 165, 166, and 167 which give the constants, represents the solution for the optimum weighting function for the case considered here. It may be noted that the solution is not a function of the initial time t_o. This is a consequence of the equatorial geometry which has been assumed.

The behavior of Equation 154 may be studied by considering a specific set of parameters $\overline{x_{10}^2}$, $\overline{\epsilon_0^2}$, and b^2. For a typical system, these parameters may be

$$\begin{aligned} \overline{\epsilon_0^2} &= 10^{-7} \; \widehat{\text{min}}^2\text{-sec}^{-2} \\ \overline{x_{10}^2} &= 5 \; \widehat{\text{min}}^2 \\ b^2 &= 5 \; \widehat{\text{min}}^2\text{-sec} \end{aligned} \tag{168}$$

The value of $\overline{\epsilon_0^2}$ chosen above corresponds to a gyro drift rate bias of approximately 0.015 degrees per hour rms. Using these parameters, one may compute the coefficients shown in Table 1. Inasmuch as $W_1(T,0)$ and $W_1(T,T)$ do not contribute to the filter output nor to the calculation of the residual error, the values of the constants C_3 and C_4 are not of significant importance. However, they are listed for the sake of rigor.

Table 1. Coefficients of $W_1(T,\tau)$

T (minutes)	C_1	C_2	C_3	C_4
1	1.639×10^{-2}	0.607×10^{-6}	0.328×10^{-2}	0.328×10^{-2}
5	0.2892×10^{-2}	0.2879×10^{-5}	0.0578×10^{-2}	1.78×10^{-3}
30	-0.9524×10^{-3}	1.6753×10^{-6}	-0.1905×10^{-3}	1.52×10^{-2}
$\to \infty$	$-2/T$	$6/T^2$		

The ensemble average error squared is given by Equation 114 which is repeated below:

$$\overline{\delta X_1^2} = \overline{X_1^2(t_o + T)} - \int_0^T W_1(T,\tau)\, \overline{X_1(t_o + T)\left[X_1(t_o + \tau) + \Delta X_1(t_o + \tau)\right]}\, d\tau \tag{114}$$

The term $\overline{X_1^2(t_o + T)}$ may be obtained using Equation 115. For the case considered here

$$\overline{X_1^2(t_o + T)} = \overline{X_{10}^2} + \overline{\epsilon_0^2} T^2 \tag{169}$$

The correlation function in the integral of Equation 114 is simply $I_1(T,\tau)$. Again for this case

$$I_1(T,\tau) = \overline{X_{10}^2} + \overline{\epsilon_0^2} T\tau \tag{170}$$

Thus Equation 114 may be written as

$$\overline{\delta X_1^2} = \overline{X_{10}^2} + \overline{\epsilon_0^2} T^2 - \int_0^T W_1(T,\tau)\left[\overline{X_{10}^2} + \overline{\epsilon_0^2} T\tau\right] d\tau \tag{171}$$

where $W_1(T,\tau)$ is given by Equation 154 with coefficients tabulated in Table 1 for typical values of T. As stated earlier, the values $W_1(T,0)$ and $W_1(T,T)$ may be chosen arbitrarily without affecting the computation of $\overline{\delta X_1^2}$. Accordingly, in Equation 171 one may use

$$W_1(T,\tau) = C_1 + C_2\tau \qquad 0 \leq \tau \leq T \tag{172}$$

Substitution of Equation 172 into Equation 171 and performing the integration gives

$$\overline{\delta\chi_1^2} = \overline{\chi_{10}^2}(1 - C_1T - \frac{C_2}{2} T^2) + \overline{\epsilon_0^2}T^2(1 - \frac{C_1}{2} T - \frac{C_2}{3} T^2) \qquad (173)$$

Using the tabulated values of C_1 and C_2 in Table 1 in Equation 173 gives the values of $\overline{\delta\chi_1^2}$ shown below in Table 2. The improvement in the estimate of χ_1 with increasing T is evident from the data.

Table 2. Computed residual error

T (minutes)	$\overline{\delta\chi_1^2}$	$(\overline{\delta\chi_1^2})^{1/2}$
1	7.77×10^{-2}	0.28 $\widehat{\text{min}}$.
5	3.46×10^{-2}	0.18 $\widehat{\text{min}}$.
30	1.73×10^{-2}	0.13 $\widehat{\text{min}}$.
$\rightarrow \infty$	$\rightarrow 0$	$\rightarrow 0$

It is interesting to compare the range of T with the drift-rate "correlation time" of practical floated gyroscopes. Typically T is of the order of minutes, whereas the correlation time of the random portion of the gyro drift rate may be several hours. Accordingly, the correlation function

$$\phi_\epsilon'(\tau) = a^2 e^{-c|\tau|}$$

which we assumed to be zero in the above analysis, may actually be treated as a constant, a^2, in the integral equation for $W_1(T,\tau)$. Thus the optimum weighting function derived above is also optimal for this more general case if we replace $\overline{\epsilon_0^2}$ by $\overline{\epsilon_0^2} + a^2$.

V. CONCLUSION

Rapid advances in the technology of inertial navigation systems have been made in recent years which have been due in part to a corresponding improvement in gyro technology. Gyroscope performance is currently the limiting factor in marine inertial navigation designs and will probably continue to be so unless an order of magnitude improvement in gyro performance can be achieved. As a result there is a real need to provide an external reference, preferably passive, which can be utilized to compensate for the long-term gyro drift characteristics. As stated earlier, star observations made either optically or radiometrically can provide this reference. Because of the limited number of radiometric celestial bodies, this study has been devoted to the development of a correction technique which utilizes a single star vector and the earth's rotation vector to uniquely determine the inertial system error vector.

The operational restrictions of the hybrid celestial-inertial system are treated in Section III and need not be discussed extensively here. Briefly, measurement of a star vector was postulated during the time interval t_0 through $t_0 + T$. At the end of the observation period, an estimate of the system error vector, $\underline{X}$, is to be made based on an optimal linear time-varying filter operating on the data obtained from the star vector during the interval T.

To accomplish this result two equations must be mechanized. The first, Equation 46, defines the error vector in terms of measured and computed components of the star vector and its rate of change relative to the inertial navigation coordinate system. This equation provides a "continuous" raw estimate of the inertial system error in the interval t_0 to $t_0 + T$.

The second equation which must be mechanized is Equation 51, the convolution integral, where $W_1'(t,\tau)$ used in Equation 51 is given implicitly by Equations 54 and 60. As stated earlier, no explicit closed-form expression for $W_1(T,\tau)$ has been obtained for the general case. However, $W_1(T,\tau)$ may be obtained approximately by numerical solution of Equation 60 for the geometries which one may encounter. The optimal filter normally cannot be realized as a lumped-parameter filter, and may prove difficult to mechanize by digital means. As a result, it is customary to approximate the optimum filter with a less than optimum device which can be implemented. In this connection, this study may prove to be of significant value in that it provides a lower bound on the residual error with which the efficiency of the approximating device can be compared. Clearly, the function $W_1(T,\tau)$ provides the characteristic which is to be approximated as well.

In executing this study certain assumptions concerning the characteristics of the error sources have of necessity been made. Perhaps the most fundamental is the assumption that the level errors in the inertial system, described by the vector components ζ_1 and ζ_2, are small. This condition is realistic for a precision system and does not materially restrict the validity of the study. An assumption which is somewhat more restrictive is inherent in the error model chosen for the basic inertial navigator. In developing the error model, and in subsequent computations, the effect of interaxis coupling due to position and azimuth errors has been neglected. This assumption does not alter the usefulness of the techniques described here, but it may affect the manner in which they are applied. Specifically, the functions $W_1(T,\tau)$ may not be optimum if there is significant interaxis coupling. If the system has been operating in its pure inertial mode for an extended period of time so that pronounced interaxis coupling may exist, an interim correction, based on a relatively short star observation, should be made to reduce the components of the error vector $\underline{\chi}$ below approximately five minutes of arc. Following the interim correction, the normal correction may then be employed.

In the derivation of the correlation functions of Section III.C.1, the gyro drift-rate error has been assumed to be stationary in the statistical sense. This assumption is not fundamental to the derivation and may be omitted if desired provided the corresponding correlation functions are changed accordingly. However, the stationary property is commonly used, and normally represents a reasonable approximation for precision floated gyros.

This study has been devoted principally to development of a method for forming a best estimate of the inertial system error. The method by which the system is actually corrected with the computed estimate of error, has been discussed only briefly. Fundamentally there are three methods for system correction:

1. A single correction may be applied at time $t_o + T$, which has been computed from star data taken over the interval of duration T. In this method a single filter characteristic, optimum with respect to the parameters t_o, T and the geometry, operates on the raw error data over the interval T to provide the optimum estimate at time $t_o + T$.
2. Multiple corrections may be applied at times t_1, t_2, $\cdots$, t_n to provide a semi-continuous correction. The ith correction is computed from data taken over the interval from $t_i - t_{i-1}$ and the ith filter characteristic is chosen on the basis of the parameters t_{i-1} for the initial time and $t_i - t_{i-1}$ for the observation interval. If the total elapsed time between t_o and t_n is T, then this method is less

accurate than method 1 for all t_i since in general each estimate is made on the basis of an observation time less than T.

3. If the raw estimate of the system error can be stored as a semi-continuous function of time, or if n filters are simultaneously available, multiple corrections can be applied as in method 2 without the resultant increase in residual error. The ith correction is computed from data taken over the interval $t_i - t_o$ and the ith filter characteristic is chosen with respect to that interval and t_o. The residual error at time $t_o + T$ is the same as that for method 1. It should be noted, however, that after any correction has been applied, all subsequent data must be compensated in accordance with the applied correction.

The choice of the most appropriate method is dependent on the system operational requirements. The first method is clearly the simplest to mechanize and should be adequate for most applications.

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VII. ACKNOWLEDGEMENT

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VIII. APPENDIX A: RELATION BETWEEN THE UNIT VECTORS OF TWO NEARLY-COINCIDENT COORDINATE SYSTEMS

Consider two orthogonal coordinate systems denoted by y and z with unit vectors $\underline{y}_1$, $\underline{y}_2$, $\underline{y}_3$ and $\underline{z}_1$, $\underline{z}_2$, $\underline{z}_3$ respectively (see Figure A-1). If the angle between each pair of unit vectors $\underline{y}_i$ and $\underline{z}_i$ is small, the vectors $\underline{z}_i$ may be expressed in terms of the vectors $\underline{y}_i$ and a rotation vector $\underline{\zeta}$ which describes the small rotations about the $\underline{y}_i$ axes which are necessary to rotate the y-coordinate system into the z-coordinate system. Let $\underline{\zeta}$ be defined as

$$\underline{\zeta} = \zeta_1\underline{y}_1 + \zeta_2\underline{y}_2 + \zeta_3\underline{y}_3 \qquad \text{(A-1)}$$

where ζ_i denotes a small rotation about the $\underline{y}_i$ axis. From Figure A-1 it is seen that

$$\underline{z}_1 = \underline{y}_1 + \zeta_3\underline{y}_2 - \zeta_2\underline{y}_3 \qquad \text{(A-2)}$$

Similarly

$$\underline{z}_2 = \underline{y}_2 + \zeta_1\underline{y}_3 - \zeta_3\underline{y}_1 \qquad \text{(A-3)}$$

$$\underline{z}_3 = \underline{y}_3 + \zeta_2\underline{y}_1 - \zeta_1\underline{y}_2 \qquad \text{(A-4)}$$

Expressions A-2, A-3, and A-4 may be written as

$$\underline{z}_i = \underline{y}_i + \underline{\zeta} \times \underline{y}_i \qquad \text{(A-5)}$$

In Sections II and III, similar expressions for an arbi-

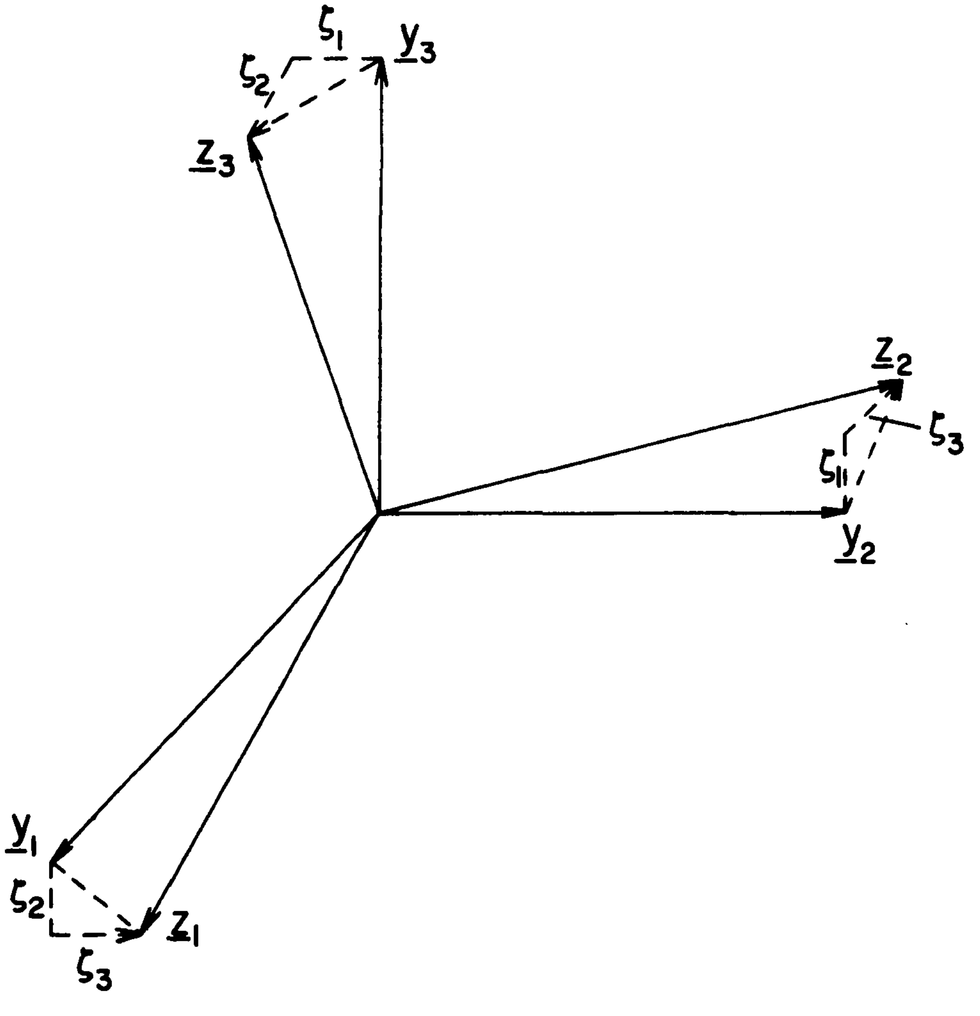

FIGURE A-1. VECTOR ROTATIONS OF NEARLY COINCIDENT COORDINATE SYSTEMS.

trary vector $\underline{V}$ are required. Let $\underline{V}_z$ be a vector whose components in the z-coordinate system are given by the relation

$$\underline{V}_z = V_{z_1}\underline{z}_1 + V_{z_2}\underline{z}_2 + V_{z_3}\underline{z}_3 \qquad \text{(A-6)}$$

Let $\underline{V}_y$ be defined as the vector whose components in the y-coordinate system are equal to the components of $\underline{V}_z$ in the z-coordinate system, i.e.,

$$\underline{V}_y = V_{z_1}\underline{y}_1 + V_{z_2}\underline{y}_2 + V_{z_3}\underline{y}_3 \qquad \text{(A-7)}$$

Using Equation A-5, Equation A-7 becomes

$$\underline{V}_y = V_{z_1}(\underline{z}_1 - \underline{\zeta} \times \underline{y}_1) + V_{z_2}(\underline{z}_1 - \underline{\zeta} \times \underline{y}_2) + V_{z_3}(\underline{z}_3 - \underline{\zeta} \times \underline{y}_3) \qquad \text{(A-8)}$$

For small rctations

$$\underline{\zeta} \times \underline{y}_i \cong \underline{\zeta} \times \underline{z}_i \qquad \text{(A-9)}$$

Therefore Equation A-8 may be written as

$$\underline{V}_y = \underline{V}_z - \underline{\zeta} \times \underline{V}_z \qquad \text{(A-10)}$$

IX. APPENDIX B: SUFFICIENCY TEST OF THE INTEGRAL EQUATION FOR $W_1(T,\tau)$

In Section III, the necessary condition which $W_1(T,\tau)$ must satisfy for the residual ensemble average error squared to be minimized was shown to be

$$\int_0^T W_1(T,\tau)\overline{\left[\chi_1(t_o + \tau_1) + \Delta\chi_1(t_o + \tau_1)\right] \times \left[\chi_1(t_o + \tau) + \Delta\chi_1(t_o + \tau)\right]}d\tau_1 - \overline{\chi_1(t_o + T)\left[\chi_1(t_o + \tau) + \Delta\chi_1(t_o + \tau)\right]} = 0 \qquad (60)$$

This condition is also sufficient as may be shown by replacing $W_1(T,\tau)$ by another function

$$W_1'(T,\tau) = W_1(T,\tau) + aK(T,\tau) \qquad \text{(B-1)}$$

where $K(T,\tau)$ may be any arbitrary function of T and τ. The ensemble average error squared is given by Equation 56 with $W_1(T,\tau)$ replaced by $W_1'(T,\tau)$ of Equation B-1, above

$$\overline{\delta\chi_1^2(a)} = \int_0^T\int_0^T\left[W_1(T,\tau_1)+aK(T,\tau_1)\right]\left[W_1(T,\tau_2)+aK(T,\tau_2)\right] \times \overline{\left[\chi_1(t_o + \tau_1) + \Delta\chi_1(t_o + \tau_1)\right] \times \left[\chi_1(t_o + \tau_2) + \Delta\chi_1(t_o + \tau_2)\right]}d\tau_1 d\tau_2 + \overline{\chi_1^2(t_o + T)} - 2\int_0^T\left[W_1(T,\tau)+aK(T,\tau)\right]\overline{\chi_1(t_o+T) \times} \qquad \text{(B-2)}$$

$$\overline{\left[X_1(t_o + \tau) + \Delta X_1(t_o + \tau) \right]} d\tau$$

Using Equation 56, Equation B-2 may be written as

$$\overline{\delta X_1^2(a)} = \overline{\delta X_1^2(0)} + 2a\int_0^T d\tau \; K(T,\tau) \left\{ \int_0^T W_1(T,\tau_1) \; x \right.$$
$$\overline{\left[X_1(t_o+\tau_1)+\Delta X_1(t_o+\tau_1) \right] \left[X_1(t_o+\tau)+\Delta X_1(t_o+\tau) \right]} d\tau_1$$
$$\left. - \overline{X_1(t_o + T) \left[X_1(t_o + \tau) + \Delta X_1(t_o + \tau) \right]} \right\} \qquad \text{(B-3)}$$
$$+ \; a^2 \overline{\left\{ \int_0^T K(T,\tau) \left[X_1(t_o + \tau) + \Delta X_1(t_o + \tau) \right] d\tau \right\}^2}$$

If $W_1(T,\tau)$ satisfies Equation 60, it is seen that the second term of Equation B-3 is zero. Accordingly, Equation B-3 becomes

$$\overline{\delta X_1^2(a)} = \overline{\delta X_1^2(0)} + a^2 \; \overline{\int_0^T K(T,\tau) \left[X_1(t_o+\tau)+\Delta X_1(t_o+\tau) \right] d\tau \quad ^2} \qquad \text{(B-4)}$$

The last term of Equation B-4 is the ensemble average of a squared quantity and is always greater than or equal to zero. Thus

$$\overline{\delta X_1^2(a)} \geq \overline{\delta X_1^2(0)} \qquad \text{(B-5)}$$

for any arbitrary function $aK(T,\tau)$.

Accordingly, Equation 60 is a necessary and sufficient condition for $W_1(T,\tau)$ to produce the minimum ensemble average residual error squared.

X. APPENDIX C: DERIVATION OF THE EQUATIONS OF MOTION IN A LOCALLY LEVEL COORDINATE SYSTEM[1]

Let $\underline{r}$ be the position vector of a point in space measured in an inertial frame of reference with origin at the center of the earth and with one axis colinear with the earth's polar axis. Let z denote the locally level set of axes with origin at $\underline{R}$ relative to the z-axes. The position vector of the point in space shall be denoted by $\underline{\rho}$. (See Figure C-1.)

Let the derivative with respect to inertial space be denoted by $\frac{d}{dt}$ and the derivative with respect to the z-coordinate system by $\frac{D}{dt}$. Accordingly, we may write

$$\frac{d\underline{r}}{dt} = \frac{d\underline{\rho}}{dt} + \frac{d\underline{R}}{dt} \qquad \text{(C-1)}$$

Using the Theorem of Coriolis, Equation C-1 may be written as

$$\frac{d\underline{r}}{dt} = \frac{d\underline{R}}{dt} + \frac{D\underline{\rho}}{dt} + \underline{\omega} \times \underline{\rho} \qquad \text{(C-2)}$$

In Equation C-2, $\underline{\omega}$ is the angular rotation vector of z-coordinate system relative to the earth-centered coordinate system. Differentiating once more and again applying the Theorem of Coriolis, gives

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[1]This derivation is similar to those found in many references: see for example Lass (3).

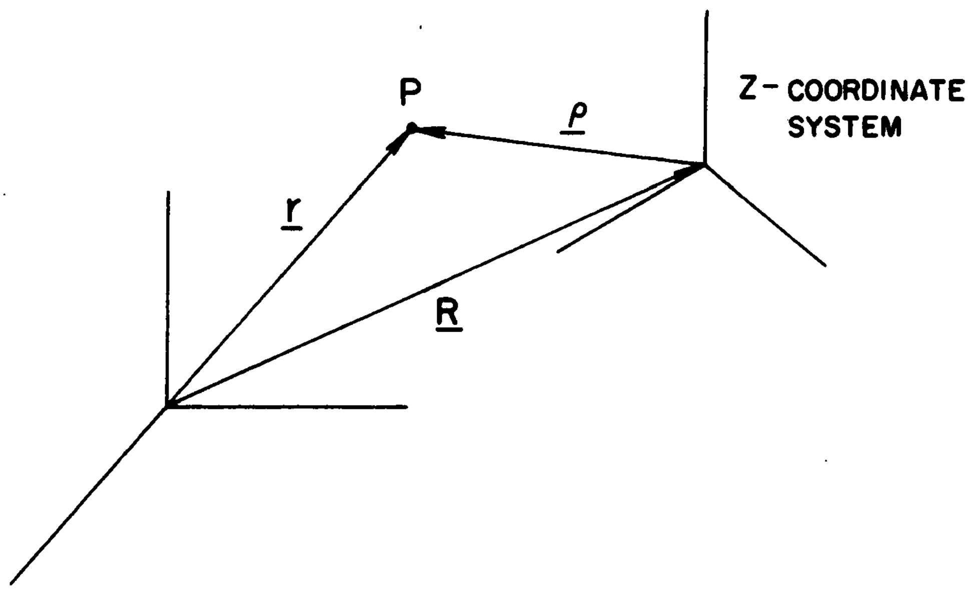

FIGURE C-1. VECTOR RELATIONS USED IN EQUATION C-1.

$$\frac{d^2\underline{r}}{dt^2} = \frac{d^2\underline{R}}{dt^2} + \frac{D^2\underline{\rho}}{dt^2} + \frac{D\underline{\omega}}{dt} \times \underline{\rho} + 2\underline{\omega} \times \frac{D\underline{\rho}}{dt} + \underline{\omega} \times (\underline{\omega} \times \underline{\rho}) \qquad \text{(C-3)}$$

It should be noted that $\frac{D\underline{\omega}}{dt} = \frac{d\underline{\omega}}{dt}$. Now, if $\underline{r} = 0$ and $\underline{\rho} = -\underline{R}$, Equation C-3 may be written as

$$\frac{d^2\underline{R}}{dt^2} = \frac{D^2\underline{R}}{dt^2} + \frac{D\underline{\omega}}{dt} \times \underline{R} + 2\underline{\omega} \times \frac{D\underline{R}}{dt} + \underline{\omega} \times (\underline{\omega} \times \underline{R}) \qquad \text{(C-4)}$$

Accelerometers measure the vector, $\underline{A} = \frac{d^2\underline{R}}{dt^2} - \underline{G}$ so that Equation C-4 may be written

$$\underline{A} + \underline{G} = \frac{D^2\underline{R}}{dt^2} + \frac{D\underline{\omega}}{dt} \times \underline{R} + 2\underline{\omega} \times \frac{D\underline{R}}{dt} + \underline{\omega} \times (\underline{\omega} \times \underline{R}) \qquad \text{(C-5)}$$

where $\underline{A}$ represents the vector output of the accelerometers, and $\underline{G}$ is the gravity vector.

Assuming that the earth is spherically symmetrical, a first order approximation for $\underline{G}$ is

$$\underline{G} = -\frac{g}{r_o}\underline{R} \qquad \text{(C-6)}$$

where r_o is the radius of the earth and g is the gravitational field constant. Thus Equation C-5 becomes

$$\underline{A} - \frac{g}{r_o}\underline{R} = \frac{D^2\underline{R}}{dt^2} + \frac{D\underline{\omega}}{dt} \times \underline{R} + 2\underline{\omega} \times \frac{D\underline{R}}{dt} + \underline{\omega} \times (\underline{\omega} \times \underline{R}) \qquad \text{(C-7)}$$

If it is assumed that the z-coordinate system is locally level with axes $\underline{z}_1$, $\underline{z}_2$, and $\underline{z}_3$ north, west, and vertical, the following expression for $\underline{R}$ and its derivatives are valid

$$\underline{R} = r_0 \underline{z}_3 \tag{C-8}$$

$$\frac{D\underline{R}}{dt} = \frac{D^2\underline{R}}{dt^2} = 0 \tag{C-9}$$

Also, the angular rotation vector $\underline{\omega}$ written as a column vector is as follows

$$\omega = \begin{pmatrix} \omega_{z_1} \\ \omega_{z_2} \\ \omega_{z_3} \end{pmatrix} = \begin{pmatrix} (\Omega - \dot{\lambda})\cos\theta \\ \dot{\theta} \\ (\Omega - \dot{\lambda})\sin\theta \end{pmatrix} \tag{C-10}$$

In Equation C-10, Ω is the scaler magnitude of the earth's rotation rate, λ represents longitude and θ represents latitude. Substitution of Equations C-8, C-9 and C-10 into Equation C-7 gives

$$\begin{pmatrix} A_{z_1} \\ A_{z_2} \\ A_{z_3} \end{pmatrix} = \begin{pmatrix} r_0\ddot{\theta} \\ r_0\left[(\Omega - \dot{\lambda})\dot{\theta}\sin\theta + \ddot{\lambda}\cos\theta\right] \\ 0 \end{pmatrix} + \begin{pmatrix} r_0(\Omega - \dot{\lambda})^2\sin\theta\cos\theta \\ r_0\dot{\theta}(\Omega - \dot{\lambda})\sin\theta \\ -r_0\dot{\theta}^2 - r_0(\Omega - \dot{\lambda})^2\cos^2\theta \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix} \tag{C-11}$$

Only the first two components of Equation C-11 need to be mechanized. Solution of Equation C-11 for θ and λ by double integration of the components $r_0\ddot{\theta}$ and $r_0\ddot{\lambda}\cos\theta$ give the position in terms of the input to the system, A_{z_1} and A_{z_2}, and the coordinates of the initial position.

XI. APPENDIX D: COMPUTATION OF CORRELATION TERMS RELATED TO GYRO ERRORS

The integral equation for the optimum weighting function, Equation 60, and the equation for the residual error, Equation 114, both have terms of the form

$$h_1 = \int_0^{\tau} d\beta_2 \int_0^{\tau_1} [\overline{\epsilon_0^2} + \phi'_{\epsilon}(\beta_1 - \beta_2)] d\beta_1 \qquad \text{(D-1)}$$

and

$$h_2 = \int_0^{\tau_1} [\overline{\epsilon_0^2} + \phi'_{\epsilon}(\beta - \tau)] d\beta \qquad \text{(D-2)}$$

Since these terms are required in the computation of Section IV, the integrals D-1 and D-2 shall be evaluated assuming

$$\phi'_{\epsilon}(\tau) = a^2 e^{-c|\tau|} \qquad \text{(D-3)}$$

The form of ϕ'_{ϵ} given by Equation D-3 is a reasonably good approximation for the correlation function of the random components of gyro drift rate for most precision single-degree-of-freedom gyroscopes. To compute h_1, Equation D-3 may be substituted in Equation D-1 to give

$$h_1 = \int_0^{\tau} d\beta_2 \int_0^{\tau_1} \left[\overline{\epsilon_0^2} + a^2 e^{-c|\beta_1 - \beta_2|}\right] d\beta_1 \qquad \text{(D-4)}$$

Two cases must be considered:

Case 1: $\tau_1 \geq \tau$

For this case the integral may be expanded as

$$h_1 = \int_0^{\tau} d\beta_2 \int_0^{\tau_1} \overline{\epsilon_0^2}\, d\beta_1 + a^2 \int_0^{\tau} d\beta_1 \int_0^{\beta_1} e^{-c(\beta_1-\beta_2)}\, d\beta_2$$

$$+ a^2 \int_0^{\tau} d\beta_2 \int_0^{\beta_2} e^{-c(\beta_2-\beta_1)}\, d\beta_1 + a^2 \int_{\tau}^{\tau_1} d\beta_1 \int_0^{\tau} e^{-c(\beta_1-\beta_2)}\, d\beta_2 \qquad \text{(D-5)}$$

Noting the symmetry of the second and third integrals, and performing the integrations gives

$$h_1 = \overline{\epsilon_0^2}\,\tau_1 \tau + \frac{2a^2}{c}\,\tau + \frac{a^2}{c^2}\left[e^{-c\tau_1}(1 - e^{c\tau}) - (1 - e^{-c\tau}) \right] \qquad \text{(D-6)}$$

for $\tau_1 \geq \tau$.

Case 2: $\tau_1 \leq \tau$

Expanding the integral in a manner similar to that of Case 1 and integrating gives

$$h_1 = \overline{\epsilon_0^2}\,\tau_1 \tau + \frac{2a^2}{c}\,\tau_1 + \frac{a^2}{c^2}\left[e^{-c\tau}(1 - e^{c\tau_1}) - (1 - e^{-c\tau_1}) \right] \qquad \text{(D-7)}$$

for $\tau_1 \leq \tau$.

Two cases must also be treated in evaluating the integral h_2 given by Equation D-2. Substitution of Equation D-3 in D-2 gives

$$h_2 = \int_0^{\tau_1} \left[\overline{\epsilon_0^2} + \phi'_{\epsilon}(\beta - \tau) \right] d\beta \qquad \text{(D-8)}$$

Case 1:

In this case the integral may be expanded as

$$h_2 = \int_0^{\tau_1} \overline{\epsilon_0^2} d\beta + a^2 \int_0^{\tau} e^{-c(\tau-\beta)} d\beta + a^2 \int_{\tau}^{\tau_1} e^{-c(\beta-\tau)} d\beta \quad \text{(D-9)}$$

Performing the integration gives

$$h_2 = \overline{\epsilon_0^2}\tau_1 + \frac{a^2}{c}(1 - e^{-c\tau}) + \frac{a^2}{c}(1 - e^{-c(\tau_1-\tau)}) \quad \text{(D-10)}$$

for $\tau_1 \geq \tau$.

Case 2: $\tau_1 \leq \tau$

Expansion of Equation D-9 gives

$$h_2 = \int_0^{\tau_1} \overline{\epsilon_0^2} d\beta + a^2 \int_0^{\tau_1} e^{-c(\tau-\beta)} d\beta \quad \text{(D-11)}$$

After integration, Equation D-11 becomes

$$h_2 = \overline{\epsilon_0^2}\tau_1 + \frac{a^2}{c} e^{-c\tau}(e^{c\tau_1} - 1) \quad \text{(D-12)}$$

for $\tau_1 \leq \tau$.